PART I: EXCITONS AND PLASMONS IN SUPERCONDUCTORS

PART II: LIFETIME EFFECTS IN CONDENSED FERMION SYSTEMS

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Technical Report No. 1

November, 1961

SOLID STATE SCIENCES DIVISION
Air Force Office of Scientific Research
Washington 25, D. C.
Contract No. AF 49(638)-882
J. R. Schrieffer, Principal Investigator

This report consists of a reproduction of part of the thesis submitted by Angelo Bardasis to the University of Illinois in partial fulfillment of the requirements for the degree, Doctor of Philosophy. The work described herein received partial support from the Air Force Office of Scientific Research under Contract AF 49(638)-882 and was supervised by Professor J. R. Schrieffer.

ERRATA

PART I

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11 — Eq. (2.9a) reads $u_k = \left[\frac{1}{2}(1 + \epsilon_k/E_k)\right]^{\frac{1}{2}}$

11 — Eq. (2.9b) reads $v_k = \left[\frac{1}{2}(1 + \epsilon_k/E_k)\right]^{\frac{1}{2}}$

35 — Second line of Eq. (3.19b) reads

"... $\delta_{L2}\delta_{M0}\Big] Z(\vec{q})$."

57 — Line 4 reads "... ɹrrent density the last term"

58 — Eq. (5.21) reads

$$= 2\pi(\hbar) \int_0^\infty \mathrm{Im}\left[K_{coll}(\vec{q},\omega)\, \vec{a}^2(\vec{q},\omega)\right] dq \ .$$

59 — Eq. (5.22) reads

$$R_s(\omega) = \frac{2(\hbar\omega)}{\pi} \lambda_L^4 \int_0^\infty \mathrm{Im}\ K_{coll}(\vec{q},\omega)\ dq \ .$$

81 — Reference 14 is A195, 336 (1949).

PART II

86 — Line 11 reads "acting within the energy band..."

91 — Eq. (2.6b) reads $G_p^{\,0} = \dfrac{1}{p_0 - (\epsilon_p - i\eta_p)\tau_3}$,

91 — Eq. (2.7) reads

$$G_p^{-1} = (G_p^{\,0})^{-1} - \Sigma_p \ .$$

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92 The denominator of Eq. (2.10) reads

$$p_0^2 - \frac{1}{Z_p^2}\left[(\epsilon_p + \chi_p)^2 + \Delta_p^2\right] + i\eta$$

93 The denominator of the integrand of Eq. (2.13) reads

$$(p_0')^2 - \frac{1}{Z_{p'}^2}\left[(\epsilon_{p'} + \chi_{p'})^2 + \Delta_{p'}^2\right] + i\eta$$

93 Line 8 reads "Therefore, the $i\eta$ term may..."

104 Reference 12 is (The Physical Society, London, 1959) 22.

ACKNOWLEDGMENTS

The author deeply thanks Dr. J. Robert Schrieffer for suggesting these problems and for the sincere interest and understanding with which he supervised their solutions. He also thanks Dr. John Bardeen for the clarification and direction he gave regarding several difficulties which arose during the course of this work.

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Part I

Excitons and Plasmons in Superconductors

I. INTRODUCTION

In the original theory of Bardeen, Cooper, and Schrieffer[1] an approximation to the ground-state wave function of a superconductor was obtained by a variational calculation. Basic to the theory is Cooper's result[2] that if a net attraction exists between the particles, the Fermi sea is unstable with respect to the formation of bound pairs. The BCS ground-state wave function is formed from a linear combination of normal state-like configurations in which particles are excited to states of low energy above the Fermi surface. In all of these normal configurations, the single-particle states are occupied in pairs $(\vec{k}\uparrow, -\vec{k}\downarrow)$, so that interactions other than those between pairs of electrons of zero net momentum and spin are neglected. The theory leads to the single quasi-particle excitation spectrum given by $E_k = (\epsilon_k^2 + \Delta_k^2)^{1/2}$, where ϵ_k is the Bloch energy measured with respect to the Fermi level and Δ_k is the energy gap; that is, $2\Delta_k$ represents the minimum energy required to excite a pair of quasi-particles from the ground state. The quasi-particle excitations are fermions and no boson excitations appear other than the phonons.

This independent quasi-particle approximation has been surprisingly successful in explaining the thermodynamic properties as well as the acoustic and electromagnetic absorption, the nuclear spin relaxation, and the Meissner effect observed in the superconducting state. The derivation of the last has been criticized because it

is not strictly gauge-invariant. This is primarily due to the neglec of residual interactions between particles in states $-\vec{k}$ and $\vec{k}' \neq \vec{k}$. These interactions give rise to a set of collective excitations (bosons) and lead to a gauge-invariant description of the Meissner effect.

For the investigation of these collective excitations, Anderson[3] and Bogoliubov, Tolmachev, and Shirkov[4] have used a generalized time-dependent self-consistent field or random-phase approximation (RPA). Their work shows that in the superconducting state, the plasmon frequency and the plasmon coordinate in the long-wavelength limit are essentially the same as in the normal state. They have also suggested the existence of the exciton modes lying within the energy gap which we investigate in the main body of this paper. A thorough discussion of the generalized RPA has been given by Rickayzen,[5] who used it to derive the complex dielectric constant of a superconductor and the Meissner effect in a gauge-invariant manner. The BCS quasi-particle states $|\alpha\rangle$ and $|\beta\rangle$ do not satisfy the continuity equation; that is, $\langle\alpha| \nabla\cdot\vec{J} + \dot{\rho}|\beta\rangle \neq 0$. When collective modes are included, the current and charge density operators $\vec{J}$ and ρ are decomposed into a sum of individual-particle operators and collective operators. A virtual cloud of plasmons surrounds each quasi-particle, producing a back-flow current which leads to over-all charge conservation of the excitation. Therefore, the continuity equation is satisfied within the generalized RPA. This condition is sufficient to guarantee a gauge-invariant form of the electromagnetic response kernel.

In this thesis we interpret the exciton mode in the superconductor as a bound pair of quasi-particles whose center-of-mass $\left[(\vec{r}_1 + \vec{r}_2)/2)\right]$ propagates with momentum $\hbar\vec{q}$. The exciton spectrum is investigated through the generalized RPA equations of motion proposed by Anderson in the form introduced by Rickayzen involving the quasi-particle operators γ_k of Bogoliubov[8] and Valatin[9] rather than c_k, the usual electron operators. In these equations we make an expansion of the interaction potential $V(\vec{k},\vec{k}')$ in terms of spherical harmonics. It is found that excitons may be characterized by the approximate quantum numbers L and M describing the symmetry of the states with respect to the relative coordinate $\vec{r}_1 - \vec{r}_2$. The existence of an L-state exciton (corresponding to the p,d,f,... excitons) is dependent on V_L being negative, where V_L is the L-wave part of $V(\vec{k},\vec{k}')$. The plasmon state corresponds to an s-state exciton whose energy is greatly increased by the long-range Coulomb interaction.

To obtain solutions to the Anderson-Rickayzen equations, we take matrix elements of the equations between a state with one collective excitation and the ground state which has been renormalized so as to include the zero-point motion of the collective modes. The results give two sets of solutions $\Lambda_{LM}(\vec{q})$ and $\Gamma_{LM}(\vec{q})$ which correspond to what Anderson has termed odd and even solutions. We show that the $\Lambda_{LM}(\vec{q})$ modes are unphysical and that the $\Gamma_{LM}(\vec{q})$ modes correspond to the exciton states. The quantum numbers L and M are found to be exact in the limit of zero center-of-mass momentum $\hbar\vec{q}$. For larger $\vec{q}$, states of different L are mixed, although the mixing is small for

$q\xi_0 \ll 1$, where ξ_0 is the coherence length. The magnetic quantum number M, however, remains a good quantum number for all $\vec{q}$ if the potential has no crystalline anisotropy. The exciton energy for the $\vec{q} = 0$ case is plotted as a function of the L-wave coupling constant g_L defined by $g_L = - N(0)V_L/4\pi$, where N(0) is the density of states in the normal phase at the Fermi surface. For $g_L > g_0$, the excitation energy proves to be imaginary and the implications of this with respect to the original BCS ground state are discussed. The $M \neq 0$ excitons may be considered as transverse collective excitations since they do not couple with a longitudinal field. In the general case, if the ground state is formed from L_0,M_0 pairs, the L_0,M_0 exciton becomes the plasma oscillation.

In Sec. II we discuss the generalized RPA from a diagrammatic point of view. Solutions for the collective excitations are obtained in Sec. III.

Tsuneto[6] has applied Rickayzen's analysis to the problem of the surface impedance. While he finds the existence of a precursor absorption for frequencies within the gap, his results give an absorption due to the exciton states which is an order of magnitude less than that observed by Ginsberg, Richards and Tinkham[7] in lead and mercury.

In Sec. IV we consider corrections to the Anderson-Rickayzen equations which lead to a new type of exciton of a particle-hole nature closely related to exciton states occurring in insulators. A calculation similar to Tsuneto's is performed for the electromagnetic absorption due to these new exciton states in Sec. V. As in Tsuneto's work, the ratio of the surface resistance due to excitons to that of normal metals in the extreme anomalous limit turns out to be about an order of magnitude too small to explain the observed data.

II. EQUATIONS OF MOTION

We consider a system of electrons interacting via an effective two-body potential V, whose matrix elements in the Bloch state representation are given by

$$(k_1', k_2' | V | k_1, k_2) = \tfrac{1}{2}\{V(k_1, k_1') + V(k_2, k_2')\} \times \delta_{k_1+k_2,\, k_1'+k_2'} . \tag{2.1}$$

This potential arises from both Coulomb and phonon interactions between electrons and will be discussed in detail below. The Hamiltonian is expressed in the Heisenberg representation in terms

of the operators $c^{\dagger}_{k\sigma}$ and $c_{k\sigma}$ which create and annihilate electrons in Bloch states of momentum k and spin σ. They satisfy the usual Fermi anticommutation relations. The single-particle Bloch energies ϵ_k, measured relative to the Fermi energy E_F, are assumed to be of the form $(\hbar^2k^2/2m) - E_F$. The Hamiltonian of the system is given by

$$H = \sum \epsilon_k c^{\dagger}_{k\sigma} c_{k\sigma} + \frac{1}{2} \sum_{k,k',q,\sigma,\sigma'} V(\vec{k}, \vec{k}+\vec{q}) \times c^{\dagger}_{k+q,\sigma} c^{\dagger}_{k'-q,\sigma'} c_{k',\sigma'} c_{k,\sigma} . \tag{2.2}$$

In the generalized RPA one studies the time evolution of bilinear operators of the form

$$b^{\dagger}_{k}(\vec{q}) = c^{\dagger}_{k+q\uparrow} c^{\dagger}_{-k\downarrow}, \tag{2.3a}$$

$$b_{k+q}(-\vec{q}) = c_{-k-q\downarrow} c_{k\uparrow}, \tag{2.3b}$$

$$\rho_{k\sigma}(\vec{q}) = c^{\dagger}_{k+q,\sigma} c_{k,\sigma}, \tag{2.3c}$$

which create excitations with a fixed total momentum $\hbar\vec{q}$. It is helpful to consider the full-time development of these operators as being built up from the infinitesimal change of the operators in a time interval δt; for example,

$$\delta b_k^+(\vec{q},t) = b_k^+(\vec{q},t+\delta t) - b_k^+(\vec{q},t)$$
$$= (1/i\hbar)[H, b_k^+(\vec{q},t)]\,\delta t. \qquad (2.4)$$

In the absence of the interaction V, the commutator reduces to $(\epsilon_{k+q} - \epsilon_k)b_k^+(\vec{q},t)$ so that except for a phase factor, the operators are independent of time. We call any operator μ_α^+ an eigenoperator if its time dependence is given simply by a phase factor. The equation of motion

$$[H, \mu_\alpha^+] = \hbar\Omega_\alpha \mu_\alpha^+ \qquad (2.5)$$

for the operator guarantees that μ_α^+, when applied to an eigenstate $|\beta\rangle$ of H, creates an eigenstate $|\alpha\rangle$ of H with an excitation energy $\hbar\Omega_\alpha$. From the Hermitian conjugate of (2.5) it follows that μ_α has the inverse effect of μ_α^+. That is, while μ_α^+ adds energy to the system, μ_α subtracts energy, so that μ_α^+ and μ_α may be thought of as creation and annihilation operators of excitations of the system. A knowledge of the eigenoperators and their eigenenergies allows one to calculate dynamic properties of the system as well as the thermodynamic functions.

In certain cases the state $\mu_\alpha^+|\beta\rangle$ may vanish identically; for example, if μ_α^+ creates pairs of fermions in states already occupied in $|\beta\rangle$. Another example is if the operator μ_α^+ scatters excitations already present in the initial state, in which case μ_α^+ vanishes when applied to the ground state. Both cases will be dealt with in the next section.

In the presence of the interaction V, the commutator (2.4) is complicated by the presence of terms involving four single-particle operators (c and $c^\dagger$'s). Therefore, the bilinear operators $b^\dagger$, b, and ρ are no longer eigenoperators of H and one must include products of four, six, ..., etc., single operators to form the $\mu_\alpha^\dagger$'s in this case. The question arises whether there is a consistent approximation in which the eigenoperators are represented as linear combinations of the bilinear operators $b^\dagger$, b, and ρ alone. Consider a typical term in the commutator arising from the interaction potential

$$\tfrac{1}{2} V(\vec{p},\vec{p}+\vec{q}')\left[c^\dagger_{p+q'\uparrow} c^\dagger_{-p'-q'\downarrow} c_{-p'\downarrow} c_{p\uparrow},\ b^\dagger_k(\vec{q})\right]$$

$$= \tfrac{1}{2} V(\vec{p},\vec{p}+\vec{q}')\left\{c^\dagger_{k+q+q'\uparrow} c^\dagger_{-p'-q'\downarrow} c_{-p'\downarrow} c^\dagger_{-k\downarrow}\,\delta_{p,k+q}\right.$$

$$\left. - c^\dagger_{k+q\uparrow} c^\dagger_{p+q'\uparrow} c^\dagger_{-k-q'\downarrow} c_{p\uparrow}\,\delta_{p',k}\right\}. \qquad (2.6)$$

This expression is shown in diagrammatic form in Fig. 1. In the diagram, time is increasing from right to left with the incoming particles in states $\vec{k}+\vec{q}\uparrow$ and $-\vec{k}\downarrow$ entering from the right. The first term on the right-hand side of (2.6) is represented by Fig. 1(a) in which the interaction, represented by a dashed line, scatters the spin-up incoming particle to $\vec{k}+\vec{q}+\vec{q}'\uparrow$, creating a particle and a hole in states $-\vec{p}'-\vec{q}'\downarrow$ and $-\vec{p}'\downarrow$, respectively. In Fig. 1(b)

the analogous process for the spin-down particle given by the second term in (2.6) is shown. If at time t = 0 a pair of single particles is excited, at time δt there is a finite probability that a particle-hole pair has been created from the background of particles in the Fermi sea, with the incoming particles scattering to new states. In the next interval of time a similar process may occur involving any of the four excitations, and in general the "bare" incoming particles will create a complicated cascade of excitations leading to a decay of the initial state. In the generalized random phase approximation one keeps only those terms in the commutator which conserve the number of excitations allowing for both forward and backward propagation in time (see below). This procedure corresponds to a linearization of the equations of motion by replacing two single-particle operators in each term by a c-number given by the expectation value of this pair of operators with respect to a fixed state. If this state is chosen to be the BCS ground state, defined by

$$|\psi_0\rangle = \prod_k [u_k + v_k b_k^+(0)]|0\rangle, \tag{2.7}$$

where $|0\rangle$ is the state with no particles present, conservation of momentum and spin leads to nonzero average values only for the operators $b_k^+(0)$, $b_k(0)$, and $\rho_{k\sigma}(0) \equiv n_{k\sigma}$. In terms of the parameters u_k and v_k, these averages are

$$\langle \psi_0 | b_k^+(0) | \psi_0 \rangle = \langle \psi_0 | b_k(0) | \psi_0 \rangle^* = u_k v_k , \tag{2.8a}$$

$$\langle \psi_0 | n_{k\sigma} | \psi_0 \rangle = v_k^2 . \tag{2.8b}$$

The parameters u_k and v_k are given by

$$u_k = + (1 + \xi_k / E_k)^{\frac{1}{2}} , \tag{2.9a}$$

$$v_k = + (1 - \xi_k / E_k)^{\frac{1}{2}} , \tag{2.9b}$$

where

$$E_k = + (\xi_k^2 + \Delta_k^2)^{\frac{1}{2}} , \tag{2.10}$$

and Δ_k satisfies

$$\Delta_k = \sum_{k'} V(\vec{k}, \vec{k}') \frac{\Delta_{k'}}{2E_{k'}} . \tag{2.11}$$

This prescription gives a unique linearization of the equations of motion since for $q \neq 0$ there is at most one pair of operators with zero total momentum and spin in each term. The terms retained within this approximation are shown in Fig. 2.

(1) As shown in Fig. 2(a), the conventional particle-particle scattering vertex arises from the first term in (2.6) when $\vec{p}' = \vec{k}$.

The factor of $\frac{1}{2}$ in front of V is cancelled by the term in the interaction with spins opposite to those in (2.6). This cancellation of the factor of $\frac{1}{2}$ occurs in each vertex.

(2) Another possibility, shown in Figs. 2(b) and 2(c), is for the scattered incoming particle to enter a bound state with the other incoming particle, the outgoing excitations being the particle-hole pair created from the sea. This possibility is allowed for in the linearization by including the finite average $\langle \Psi_0 | b_k^\dagger(0) | \Psi_0 \rangle$, which may be regarded as the amplitude for the pair to enter the $q = 0$ bound state, which is macroscopically occupied in $|\Psi_0\rangle$. Since a finite fraction of all the electrons occupy this bound state in the superconducting state (corresponding to the finite fraction of helium atoms occupying the $k = 0$ state is superfluid He^4), the small fluctuation $\sim N^{1/2}$ in the number of pairs N described by (2.7) leads to no difficulties in a large system. Notice that in Figs. 2(b) and 2(c), the incoming pair of particles is transformed into a particle-hole pair by the interaction. Therefore, $b_k^\dagger(\vec{q})$ and $\rho_{k\sigma}(\vec{q})$ are coupled in the equations of motion.

(3) In addition, there is the possibility that the scattered incoming particle enters the bound state with the particle created from the sea, leaving the hole and the other incoming particles as the outgoing excitations, as shown in Figs. 2(d) and 2(e). Due to the presence of the bound state, the incoming spin-up particle in Fig. 2(d) is transformed into a hole in the state of opposite momentum and spin. In the next instant of time the inverse process

may occur. It is clear that the equations of motion are simplified if one introduces "quasi-particle" operators $\gamma_{k\sigma}$ which are the proper linear combinations of particle and hole creation operators to account for these processes. The appropriate transformation, introduced first by Bogliubov and by Valatin, is

$$\gamma_{k0}^{\dagger} = u_k c_{k\uparrow}^{\dagger} - v_k c_{-k\downarrow}, \tag{2.12a}$$

$$\gamma_{k1}^{\dagger} = u_k c_{-k\downarrow}^{\dagger} + v_k c_{k\uparrow}. \tag{2.12b}$$

For mathematical simplicity we will follow Rickayzen by expressing the final linearized equations in terms of quasi-particle variables.

(4) The exchange contributions to the single-particle lines are shown in Figs. 2(f) and 2(g). As is well known, they lead to an anomalously low density of states at the Fermi surface in the normal metal unless a screened interaction is introduced. This point is discussed below. The exchange self-energy vertex can be accounted for, along with process (3), by the quasi-particle transformation (2.12).

(5) Finally, the unscattered incoming particle may enter the bound state with the particle created from the sea, leaving the hole and scattered particle as the outgoing excitations, as shown in Figs. 2(h) and 2(i). As in process (2), the pair of incoming particles is transformed into a particle-hole pair by the interaction. In the limit $q \to 0$, process (2) is more important than (5) in forming the

plasmon state. Since the momentum transfer is always $\hbar\vec{q}$ in the former process, the large matrix element of the Coulomb interaction $4\pi e^2/q^2$ dominates the latter vertex in which the momentum transfer $\hbar\vec{q}'$ may assume any value. Anderson and Rickayzen have neglected processes (4) and (5), suggesting that their effect is primarily to renormalize the single-particle energies and the effective interaction.

The terms occurring in the linearized equation of motion for $\rho_{k\sigma}(\vec{q})$ are shown in Fig. 3 and bear a close resemblance to those shown in Fig. 2. In the conventional RPA for the excitations in the normal state, only the polarization vertex [Fig. 3(b)] is retained. The so-called exchange scattering correction shown in Fig. 3(a), when combined with the polarization vertex, approximates the time evolution of $\rho_{k\sigma}(\vec{q})$ by graphs of the type shown in Fig. 4. In the limit $q \to 0$, the exchange correction to the plasmon frequency vanishes. Since matrix elements of the equations of motion are taken with respect to RPA eigenstates, two pairs may be spontaneously created from the vacuum and may interact with the incoming excitations as in Fig. 4. This process may be viewed as a propagation of the excitations backward in time, familiar in the Green's function formulation of the problem.

In the generalized RPA for the superconducting state the presence of the bound state gives rise to the vertices represented in Figs. 3(c), (d), (g), and (h), so that an incoming particle-hole pair can be transformed into either a pair of particles or a pair of

holes. Therefore, the operators $b_k^+(\vec{q})$ and $b_{k+q}(-\vec{q})$ are coupled by the density operator $\rho_{k\sigma}(\vec{q})$. The vertices occurring in the time development of $b_{k'}(\vec{q})$ are identical to those in Fig. 2 except that all arrows are reversed and the momentum $\vec{q}$ is replaced by $-\vec{q}$.

We turn now to the question of screening. Within the random-phase approximation to the normal state, the screened interaction line is represented in the limit of small wave-vector $\vec{q}$ by a sum of diagrams of the form shown in Fig. 5. Rickayzen has shown that the dielectric constant is essentially unaffected by the pairing correlations occurring in the superconducting state. It is easily seen that the vertices 2(b), 2(c), and 3(b) are automatically screened within the RPA through the presence of the polarization vertex [Fig. 3(b)] in the linearized equations. For example, when the vertex 2(b) is followed in time by a series of vertices 3(b), the effect is to replace the bare interaction line in 2(b) by the screened line shown in Fig. 5. Therefore in vertices 2(b), 2(c), and 3(b), the unscreened interaction V_D must be used. The potential V_D is given by

$$V_D(\vec{q}) = \frac{4\pi e^2}{q^2} + \frac{|v_q^i|^2}{\Omega^2 - (\omega_q^i)^2}, \tag{2.13}$$

where $\hbar\Omega$ is the energy of the excitation involved. Also, v_q^i is the bare electron-phonon interaction matrix element introduced by Bardeen and Pines[10] and ω_q^i is the bare phonon frequency. It is

essential, however, to introduce the interaction screened by the dynamical dielectric constant in the remaining vertices since it is impossible to replace the bare interaction line by the screened line through an iteration of vertices occurring in the linearized equations. The screened potential is of the form

$$V(\vec{k},\vec{k}+\vec{q}) = \frac{4\pi e^2}{q^2 K(\vec{q},\omega_{k,q})}, \tag{2.14a}$$

where the dynamical dielectric constant is given by

$$\begin{aligned} K(\vec{q},\omega_{k,q}) &= 1 + 4\pi\alpha_{ion}(\vec{q},\omega_{k,q}) + 4\pi\alpha_{el}(\vec{q},\omega_{k,q}) \\ &\simeq 1 - (\omega_q^i)^2/\omega_{k,q}^2 + k_s^2/q^2 . \end{aligned} \tag{2.14b}$$

Here, $\hbar\omega_{k,q} = \epsilon_{k+q} - \epsilon_k$ and k_s is the electronic screening wave number. In a more complete treatment involving coupled equations of motion for the electrons and the lattice, the energy $\hbar\omega_{k,q}$ would presumably be given in terms of the quasi-particle excitation energies.

For simplicity, we neglect the vertices shown in Figs. 2(h), 2(i), and 3(a). We also neglect the exchange self-energy correction since it simply renormalizes the single-particle energies. With these approximations, one obtains the equations first given by Anderson:

$$\begin{aligned} [H, b_k^+(\vec{q})] = (\varepsilon_k + \varepsilon_{k+q}) b_k^+(\vec{q}) + V_D(q)\rho(\vec{q})(u_k v_k + u_{k+q} v_{k+q}) + \\ \Delta_k \rho_{k\uparrow}(\vec{q}) + \Delta_{k+q}\rho_{-k-q,\downarrow}(\vec{q}) - (1 - v_k^2 - v_{k+q}^2)\sum_{k'} V(\vec{k},\vec{k}') b_{k'\uparrow}^+(\vec{q}) , \end{aligned} \tag{2.15a}$$

$$[H, b_{k+q}(-\vec{q})] = -(\varepsilon_k+\varepsilon_{k+q}) b_{k+q}(-\vec{q}) - V_D(\vec{q})\rho(\vec{q})(u_k v_k + u_{k+q} v_{k+q}) -$$

$$\Delta_k \rho_{-k-q\downarrow}(\vec{q}) - \Delta_{k+q}\rho_{k\uparrow}(\vec{q}) - (1-v_k^2-v_{k+q}^2)\sum_{k'} V(\vec{k},\vec{k}') b_{k'+q}(-\vec{q}),$$

(2.15b)

$$[H, \rho_{k\uparrow}(\vec{q})] = (\varepsilon_{k+q}-\varepsilon_k)\rho_{k\uparrow}(\vec{q}) + (v_k^2 - v_{k+q}^2) V_D(\vec{q})\rho(\vec{q}) +$$

$$\Delta_k b_k^+(\vec{q}) - \Delta_{k+q} b_{k+q}(-\vec{q}) + u_k v_k \sum_{k'} V(\vec{k},\vec{k}') b_{k'}^+(\vec{q})$$

$$- u_{k+q} v_{k+q} \sum_{k'} V(\vec{k},\vec{k}') b_{k'+q}(-\vec{q}), \qquad (2.15c)$$

$$[H, \rho_{-k-q\downarrow}(\vec{q})] = (\varepsilon_k - \varepsilon_{k+q})\rho_{-k-q\downarrow}(\vec{q}) - (v_k^2 - v_{k+q}^2) V_D(\vec{q})\rho(\vec{q})$$

$$- \Delta_k b_{k+q}(\vec{q}) + \Delta_{k+q} b_k^+(\vec{q}) + u_{k+q} v_{k+q} \sum_{k'} V(\vec{k},\vec{k}') b_{k'}^+(\vec{q})$$

$$- u_k v_k \sum_{k'} V(\vec{k},\vec{k}') b_{k'+q}(-\vec{q}). \qquad (2.15d)$$

The density operator $\rho(q)$ is given by

$$\rho(\vec{q}) = \sum_{k,\sigma} c_{k+q\sigma}^+ c_{k\sigma}.$$

As mentioned above, the equations can be considerably simplified by transforming to quasi-particle variables. The Anderson-Rickayzen equations are then:

$$[H, \gamma^{+}_{k+q,0}\gamma^{+}_{k,1}] = (E_{k+q}+E_k)\gamma^{+}_{k+q,0}\gamma^{+}_{k,1} + V_D(\vec{q})m(\vec{k},\vec{q})\rho(\vec{q}) - \tfrac{1}{2}\ell(\vec{k},\vec{q})A_k(\vec{q}) + \tfrac{1}{2}n(\vec{k},\vec{q})B_k(\vec{q}), \tag{2.16a}$$

$$[H, \gamma_{k+q,1}\gamma_{k,0}] = -(E_{k+q}+E_k)\gamma_{k+q,1}\gamma_{k,0} - V_D(\vec{q})m(\vec{k},\vec{q})\rho(\vec{q}) - \tfrac{1}{2}\ell(\vec{k},\vec{q})A_k(\vec{q}) - \tfrac{1}{2}n(\vec{k},\vec{q})B_k(\vec{q}), \tag{2.16b}$$

$$[H, \gamma^{+}_{k+q,\sigma}\gamma_{k,\sigma}] = (E_{k+q}-E_k)\gamma^{+}_{k+q,\sigma}\gamma_{k,\sigma}. \tag{2.16c}$$

The coherence factors are defined by

$$\ell(\vec{k},\vec{q}) = u_k u_{k+q} + v_k v_{k+q}, \tag{2.17a}$$

$$m(\vec{k},\vec{q}) = u_k v_{k+q} + v_k u_{k+q}, \tag{2.17b}$$

$$n(\vec{k},\vec{q}) = u_k u_{k+q} - v_k v_{k+q}, \tag{2.17c}$$

$$p(\vec{k},\vec{q}) = u_k v_{k+q} - v_k u_{k+q}, \tag{2.17d}$$

and the three collective variables are

$$A_k(\vec{q}) = -\sum_{k'} V(\vec{k},\vec{k}')\left[\, b^+_{k'}(\vec{q}) - b_{k'+q}(-\vec{q})\right]$$
$$= -\sum_{k'} V(\vec{k},\vec{k}')\left[\ell(\vec{k}',\vec{q})\left(\gamma^+_{k'+q0}\gamma^+_{k'1} - \gamma_{k'+q1}\gamma_{k'0}\right) + p(\vec{k}',\vec{q})\left(\gamma^+_{k'+q0}\gamma_{k'0} - \gamma^+_{k'1}\gamma_{k'+q1}\right)\right], \tag{2.18a}$$

$$B_k(\vec{q}) = \sum_{k'} V(\vec{k},\vec{k}')\left[b^+_{k'}(\vec{q}) + b_{k'+q}(-\vec{q})\right]$$
$$= \sum_{k'} V(\vec{k},\vec{k}')\left[n(\vec{k}',\vec{q})\left(\gamma^+_{k'+q0}\gamma^+_{k'1} + \gamma_{k'+q1}\gamma_{k'0}\right) - m(\vec{k}',\vec{q})\left(\gamma^+_{k'+q0}\gamma_{k'0} + \gamma^+_{k'1}\gamma_{k'+q1}\right)\right], \tag{2.18b}$$

$$\rho(\vec{q}) = \sum_{k',\sigma} \rho_{k'\sigma}(\vec{q})$$
$$= \sum_{k'}\left[m(\vec{k}',\vec{q})\left(\gamma^+_{k'+q0}\gamma^+_{k'1} + \gamma_{k'+q1}\gamma_{k'0}\right) + n(\vec{k}',\vec{q})\left(\gamma^+_{k'+q0}\gamma_{k'0} + \gamma^+_{k'1}\gamma_{k'+q1}\right)\right]. \tag{2.18c}$$

From (2.16c) we see that half of the normal mode operators are of the form $\gamma^\dagger_{k+q\sigma}\,\gamma_{k\sigma}$, which has the eigenvalue $E_{k+q} - E_k$. These operators describe scattering of excitations already present in the initial state and vanish when applied to the ground state. Since we will always take matrix elements of the equations of motion between

the ground state and an excited state, these quasi-particle conserving operators may be safely neglected.

III. SOLUTIONS OF EQUATIONS OF MOTION

For the analysis of the plasmon and exciton modes at temperature T = 0 we begin with the Anderson-Rickayzen equations of motion (2.16) for the pair operators $\gamma^{\dagger}_{k+q0}\,\gamma^{\dagger}_{k1}$ and $\gamma_{k+q1}\gamma_{k0}$. It must be kept in mind that the equations have been linearized with respect to the ground state involving s-state pairing between electrons of opposite spin and momentum, as our results depend critically upon this fact. The collective variables defined by (2.18) are substituted into the equations in order to obtain them in a form involving only the Bogoliubov-Valatin quasi-particle operators:

$$
\begin{aligned}
[H, \gamma^{\dagger}_{k+q0}\gamma^{\dagger}_{k1}] = {} & \nu_k(\vec{q})\,\gamma^{\dagger}_{k+q0}\gamma^{\dagger}_{k1} + \\
& V_D(\vec{q})\,m(\vec{k},\vec{q})\sum_{k'} m(\vec{k}',\vec{q})\,(\gamma^{\dagger}_{k'+q0}\gamma^{\dagger}_{k'1} + \gamma_{k'+q1}\gamma_{k'0}) + \\
& \tfrac{1}{2}\,\ell(\vec{k},\vec{q})\sum_{k'} V(\vec{k},\vec{k}')\,\ell(\vec{k}',\vec{q})\,(\gamma^{\dagger}_{k'+q0}\gamma^{\dagger}_{k'1} - \gamma_{k'+q1}\gamma_{k'0}) + \\
& \tfrac{1}{2}\,n(\vec{k},\vec{q})\sum_{k'} V(\vec{k},\vec{k}')\,n(\vec{k}',\vec{q})\,(\gamma^{\dagger}_{k'+q0}\gamma^{\dagger}_{k'1} + \gamma_{k'+q1}\gamma_{k'0}),
\end{aligned}
\tag{3.1a}
$$

$$[H, \gamma_{k+q1}\gamma_{k0}] = -V_k(\vec{q})\gamma_{k'+q1}\gamma_{k0}$$
$$-V_b(\vec{q})m(\vec{k},\vec{q})\sum_{k'} m(\vec{k}',\vec{q})(\gamma^+_{k'+q0}\gamma^+_{k'1} + \gamma_{k'+q1}\gamma_{k'0}) +$$
$$\tfrac{1}{2}\ell(\vec{k},\vec{q})\sum_{k'} V(\vec{k},\vec{k}')\ell(\vec{k}',\vec{q})(\gamma^+_{k'+q0}\gamma^+_{k'1} - \gamma_{k'+q1}\gamma_{k'0}) -$$
$$\tfrac{1}{2}h(\vec{k},\vec{q})\sum_{k'} V(\vec{k},\vec{k}')n(\vec{k}',\vec{q})(\gamma^+_{k'+q0}\gamma^+_{k'1} + \gamma_{k'+q1}\gamma_{k'0}).$$

(3.1b)

Those operators $\mu^\dagger_\alpha(\vec{q})$ are now considered which are linear combinations of the bilinear products of γ_k's and $\gamma_k^\dagger$'s appearing in the two equations of motion (3.1), and which create one elementary excitation of type α. Thus we desire

$$\mu^+_\alpha(\vec{q}) = \sum_k [\varphi_\alpha(\vec{k},\vec{q})\gamma^+_{k+q0}\gamma^+_{k1} + \chi_\alpha(\vec{k},\vec{q})\gamma_{k+q1}\gamma_{k0}],$$

(3.2)

with

$$H\mu^+_\alpha(\vec{q})|0\rangle = [\hbar\Omega_\alpha(\vec{q}) + W_0]\mu^+_\alpha(\vec{q})|0\rangle,$$

(3.3)

where $|0\rangle$ is not the original ground state of BCS, but the

renormalized ground state with $\mu_\alpha(\vec{q})\,|\,0\rangle = 0$. The quantity $\hbar\Omega_\alpha(\vec{q})$ represents the energy of the excitation created by the operator $\mu_\alpha^\dagger(\vec{q})$. The elementary excitation $\mu_\alpha^\dagger(\vec{q})$ may be any one of the three types involved in the theory: a pair of excited quasi-particles in scattering states, a plasmon, or an exciton.

From Eq. (3.3) and the discussion of Sec. II, we have

$$[H,\mu_\alpha^\dagger(\vec{q})]\,|\,0\rangle = \hbar\Omega_\alpha(\vec{q})\mu_\alpha^\dagger(\vec{q})\,|\,0\rangle \quad .$$

Since the commutator $[H,\mu_\alpha^\dagger(\vec{q})]$ is related to the time derivative of $\mu_\alpha^\dagger(\vec{q})$, the matrix element of $\mu_\alpha^\dagger(\vec{q})$ between the ground state $|\,0\rangle$ and the state $|\,1(\vec{q},\alpha)\rangle$ containing one excitation of energy $\hbar\Omega_\alpha(\vec{q})$ must have the time dependence $\exp[i\Omega_\alpha(\vec{q})t]$. Now, Eq. (3.2), expresses $\mu_\alpha^\dagger(\vec{q})$, within the RPA, as a linear combination of the bilinear products $\gamma_{k+q0}^\dagger\,\gamma_{k1}^\dagger$ and $\gamma_{k+q1}\gamma_{k0}$, so that we may write the inverse transformations as

$$\gamma_{k+q0}^\dagger\,\gamma_{k1}^\dagger = \sum_\beta [f_\beta(\vec{R},\vec{q})\mu_\beta^\dagger(\vec{q}) + \tilde{f}_\beta(\vec{R},\vec{q})\mu_\beta(-\vec{q})], \tag{3.4a}$$

$$\gamma_{k+q1}\gamma_{k0} = \sum_\beta [g_\beta(\vec{R},\vec{q})\mu_\beta^\dagger(\vec{q}) + \tilde{g}_\beta(\vec{R},\vec{q})\mu_\beta(-\vec{q})]. \tag{3.4b}$$

Taking matrix elements of Eq. (3.4) between $|\,0\rangle$ and $|\,1(\vec{q},\alpha)\rangle$ and using the orthonormality property of the excited states, we find

$$\langle 1(\vec{q},\alpha)|\gamma^{\dagger}_{k+q_0}\gamma^{\dagger}_{k_1}|0\rangle = \sum_{\beta} f_{\beta}(k,\vec{q})\langle 1(\vec{q},\alpha)|\mu^{\dagger}_{\beta}(\vec{q})|0\rangle$$

$$= f_{\alpha}(k,\vec{q})\exp\left[i\Omega_{\alpha}(\vec{q})t\right], \tag{3.5a}$$

$$\langle 1(\vec{q},\alpha)|\gamma_{k+q_1}\gamma_{k_0}|0\rangle = g_{\alpha}(k,\vec{q})\exp\left[i\Omega_{\alpha}(\vec{q})t\right]. \tag{3.5b}$$

The solution for the exciton mode dispersion relation is dependent on taking matrix elements of the equations of motion (3.1) between the states $|\,0\rangle$ and $|\,1(\vec{q},\alpha)\rangle$ and using the relations (3.5a) and (3.5b) so that we obtain a set of c-number equations. The resultant system of linear equations may then be solved for the normal mode frequencies and the transformation coefficients f and g.

By taking matrix elements of (3.1) we obtain:

$$\left[\hbar\Omega_{\alpha}(\vec{q}) - \nu_k(\vec{q})\right] f_{\alpha}(k,\vec{q}) =$$

$$V_0(\vec{q})\, m(k,\vec{q}) \sum_{k'} m(k',\vec{q})\left[f_{\alpha}(k',\vec{q}) + g_{\alpha}(k',\vec{q})\right] +$$

$$\tfrac{1}{2}\ell(k,\vec{q}) \sum_{k'} V(k,k')\,\ell(k',\vec{q})\left[f_{\alpha}(k',\vec{q}) - g_{\alpha}(k',\vec{q})\right] +$$

$$\tfrac{1}{2} n(k,\vec{q}) \sum_{k'} V(k,k')\, n(k',\vec{q})\left[f_{\alpha}(k',\vec{q}) + g_{\alpha}(k',\vec{q})\right], \tag{3.6a}$$

$$\left[\hbar\Omega_\alpha(\vec{q})+V_k(\vec{q})\right] g_\alpha(\vec{k},\vec{q}) =$$
$$-V_0(\vec{q})\, m(\vec{k},\vec{q}) \sum_{k'} m(\vec{k}',\vec{q})\left[f_\alpha(\vec{k}',\vec{q})+g_\alpha(\vec{k}',\vec{q})\right] +$$
$$\tfrac{1}{2}\, \ell(\vec{k},\vec{q}) \sum_{k'} V(\vec{k},\vec{k}')\, \ell(\vec{k}',\vec{q})\left[f_\alpha(\vec{k}',\vec{q})-g_\alpha(\vec{k}',\vec{q})\right]$$
$$-\tfrac{1}{2}\, n(\vec{k},\vec{q}) \sum_{k'} V(\vec{k},\vec{k}')\, n(\vec{k}',\vec{q})\left[f_\alpha(\vec{k}',\vec{q})+g_\alpha(\vec{k}',\vec{q})\right]. \tag{3.6b}$$

From (3.6) it is evident that an explicit form for $V(\vec{k},\vec{k}')$ must be chosen in order to proceed further. As emphasized in the foregoing, the BCS ground state about which the Anderson-Rickayzen equations have been linearized is one involving s-state pairing. Thus in the absence of crystalline anisotropy, the $q \to 0$ solutions must transform according to the irreducible representations of the full rotation group, i.e., the spherical harmonics. Because of this fact, we expand the two body potential $V(\vec{k},\vec{k}')$ in terms of spherical harmonics. The coordinate system is chosen so that $\vec{q}$ lies along the polar axis with θ and Φ the polar and azimuthal angles of the wave vector $\vec{k}'$ and Θ and Φ the analogous quantities for $\vec{k}$. If ζ is the angle between $\vec{k}'$ and $\vec{k}$, the use of the addition theorem gives

$$V(\vec{k},\vec{k}') = \sum_{\ell=0}^{\infty} \bar{V}_\ell(k,k')\, Y_{\ell 0}(\zeta) =$$
$$\sum_{\ell=0}^{\infty} V_\ell(k,k') \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta,\varphi)\, Y_{\ell m}(\Theta,\Phi), \tag{3.7}$$

where

$$V_\ell(k,k') = (4\pi/2\ell + 1)^{1/2}\, \bar{V}_\ell(k,k') \ .$$

A further approximation is made in setting $V_\ell(k,k') = V_\ell$, a nonzero constant, for $|\epsilon| < \hbar\omega_c$ and zero otherwise. The quantity $\hbar\omega_c$ is the average phonon energy of the order of the Debye energy. We define the coupling constant g_L by

$$g_L = -N(0)\, V_L/4\pi \ . \tag{3.8}$$

The BCS coupling constant is related to g_0 by

$$g_0 = -N(0)\, V_0/4\pi = N(0)\, V_{BCS} > 0 \ .$$

It is convenient to introduce three new variables

$$\Lambda_{\ell m}(\vec{q}) = \sum_k n(\vec{k},\vec{q})\, V_\ell\, Y_{\ell m}^*(\theta,\varphi) \times [f(\vec{k},\vec{q}) + g(\vec{k},\vec{q})], \tag{3.9a}$$

$$\Gamma_{\ell m}(\vec{q}) = \sum_k \ell(\vec{k},\vec{q})\, V_\ell\, Y_{\ell m}^*(\theta,\varphi) \times [f(\vec{k},\vec{q}) - g(\vec{k},\vec{q})], \tag{3.9b}$$

$$Z(\vec{q}) = V_D(\vec{q}) \sum_k m(\vec{k},\vec{q}) [f(\vec{k},\vec{q}) + g(\vec{k},\vec{q})], \tag{3.9c}$$

where the subscript α has been dropped from both sides of the equations for simplicity. Equations (3.6) then express the transformation coefficients f and g in terms of the new variables Λ, Γ, and Z. By substituting these expression into the defining relationships (3.9),

we obtain the following coupled integral equations to determine the eigenfrequency Ω (q):

$$\Lambda_{LN}(\vec{q}) = V_L \sum_k n(\vec{k},\vec{q}) \frac{1}{[\hbar\Omega(\vec{q})]^2 - \nu_k(\vec{q})^2} \times$$

$$\{ 2\nu_k(\vec{q})\, m(\vec{k},\vec{q})\, Z(\vec{q})\, Y^*_{LM}(\theta,\varphi) +$$

$$\nu_k(\vec{q})\, n(\vec{k},\vec{q}) \sum_{l,m} Y^*_{LM}(\theta,\varphi)\, Y_{lm}(\theta,\varphi)\, \Lambda_{lm}(\vec{q}) +$$

$$\hbar\Omega(\vec{q})\, l(\vec{k},\vec{q}) \sum_{l,m} Y^*_{LM}(\theta,\varphi)\, Y_{lm}(\theta,\varphi)\, \Gamma_{lm}(\vec{q}) \},$$

(3.10a)

$$\Gamma_{LM}(\vec{q}) = V_L \sum_k l(\vec{k},\vec{q}) \frac{1}{[\hbar\Omega(\vec{q})]^2 - \nu_k(\vec{q})^2} \times$$

$$\{ 2\hbar\Omega(\vec{q})\, m(\vec{k},\vec{q})\, Z(\vec{q})\, Y^*_{LM}(\theta,\varphi) +$$

$$\hbar\Omega(\vec{q})\, n(\vec{k},\vec{q}) \sum_{l,m} Y^*_{LM}(\theta,\varphi)\, Y_{lm}(\theta,\varphi)\, \Lambda_{lm}(\vec{q}) +$$

$$\nu_k(\vec{q})\, l(\vec{k},\vec{q}) \sum_{l,m} Y^*_{LM}(\theta,\varphi)\, Y_{lm}(\theta,\varphi)\, \Gamma_{lm}(\vec{q}) \},$$

(3.10b)

$$Z(\vec{q}) = V_0(\vec{q}) \sum_k m(\vec{k},\vec{q}) \frac{1}{[\hbar\Omega(\vec{q})]^2 - \nu_k(\vec{q})^2} \times$$
$$\{ 2\nu_k(\vec{q}) m(\vec{k},\vec{q}) Z(\vec{q}) +$$
$$\nu_k(\vec{q}) n(\vec{k},\vec{q}) \sum_{l,m} Y_{lm}(\theta,\varphi) \Lambda_{lm}(\vec{q}) +$$
$$\hbar\Omega(\vec{q}) l(\vec{k},\vec{q}) \sum_{l,m} Y_{lm}(\theta,\varphi) \Gamma_{lm}(\vec{q}) \}. \qquad (3.10c)$$

From these three equations it is immediately seen that one good quantum number for the description of an excitation is the magnetic quantum number M. In the sum over $\vec{k}$, the angular integration requires m = M, as the only φ-dependent quantities involved are the spherical harmonics. Thus, M is a good quantum number regardless of the center-of-mass momentum $\hbar\vec{q}$.

(1) q → 0 Case

In the case of zero center-of-mass momentum, Eqs. (3.10) give L as an additional good quantum number. This follows since neither the coherence factors nor the energy $\nu_k(\vec{q})$ of the quasi-particle pair are dependent on the polar angle in this case. The angular part of the sum Σ_k then reduces to

$$\int Y^{*}_{LM}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi)\, d\omega_k = \delta_{L\ell}\,\delta_{Mm}.$$

The sum Σ_k is converted into an integral by letting

$$\sum_k \rightarrow [v/(2\pi)^3] \int dk\, k^2 d\omega_k,$$

where the volume v of the normalization box is taken as unity. The radial integrals over k are all of the form

$$I^{0}_{ab\ldots} = \frac{1}{(2\pi)^3} \int \frac{a(k,0)\,b(k,0)\ldots}{(\hbar\Omega)^2 - \nu_k(0)^2}\, k^2 dk, \qquad (3.11)$$

where each of the quantities a,b,c, ... is one of the coherence factors, the energy $\nu_k(0)$ of the independent quasi-particles, or the excitation energy $\hbar\Omega$. The integration over the magnitude of $\vec{k}$ is replaced by an integration over the Bloch state energy ϵ_k, as measured from the Fermi surface, by setting

$$k^2 dk = (m/\hbar^2)^{3/2} (2E_F)^{1/2} d\xi = 2\pi^2 N(0)\, d\xi, \qquad (3.12)$$

where we have made the approximation of a constant density of states. The approximation leads to an error of order $\hbar\omega_c/E_F = 10^{-3}$. The integrals $I_{ab\ldots}{}^{0}$ are only performed over the region $-\hbar\omega_c < \epsilon < \hbar\omega_c$ since the potentials V_ℓ have been set equal to zero outside this energy band. Using (3.12), Eqs. (3.10a) and (3.10b) for the $q \rightarrow 0$ case are written as

$$(1 - V_L I^0_{\nu n^2}) \Lambda_{LM} - V_L I^0_{n\Omega \ell n} \Gamma_{LM}$$

$$= \lim_{\vec{q} \to 0} Z(\vec{q}) 2 V_L I^0_{\nu m n} \delta_{L0}, \tag{3.13a}$$

$$- V_L I^0_{n\Omega \ell n} \Lambda_{LM} + (1 - V_L I^0_{\nu \ell^2}) \Gamma_{LM}$$

$$= \lim_{\vec{q} \to 0} Z(\vec{q}) 2 V_L I^0_{n\Omega \ell m} \delta_{L0}. \tag{3.13b}$$

From these equations it is seen that the direct Coulomb interaction $4\pi e^2/q^2$ involved in $Z(\vec{q})$ only appears for the L = M = 0 state. It will be shown below that this state has a solution corresponding to a plasma oscillation with the usual plasmon energy

$$\hbar \Omega_p = \hbar (4\pi n e^2/m)^{1/2} \sim 10 \text{ ev}$$

and lies far above the gap $2\Delta \sim 10^{-3}$ ev. In this section only the $M \neq 0$ cases will be considered, in which the right-hand sides of Eqs. (2.13) become zero. Since the integrand of $I^0_{n\Omega \ell n}$ is odd about the Fermi surface within the constant density of states approximation, $I^0_{n\Omega \ell n}$ vanishes and there is no coupling between the Λ and Γ modes. The excitation energies for the $L \neq 0$ modes with zero center-of-mass momentum are then determined by the conditions:

$$(1 - V_L I^0_{\nu_n 2}) = 0, \qquad (\Lambda_{LM} \text{ mode}), \quad (3.14a)$$

$$(1 - V_L I_{\nu \ell^2}) = 0, \qquad (\Gamma_{LM} \text{ mode}). \quad (3.14b)$$

Setting $x = (\hbar\Omega/2\Delta) \leq 1$ in the integrals $I_{\nu_n 2^0}$ and $I_{\nu \ell 2^0}$ and using the definition (3.8) of the coupling constant g_L, Eqs. (3.13) become:

$$\left(\frac{1}{g_L} - \frac{1}{g_0}\right) = -\left(\frac{\text{Arcsin } x}{x}\right)(1-x^2)^{1/2}, \qquad (\Lambda_{LM} \text{ mode}), \quad (3.15a)$$

$$\left(\frac{1}{g_L} - \frac{1}{g_0}\right) = \frac{x \text{ Arcsin } x}{(1-x^2)^{1/2}}, \qquad (\Gamma_{LM} \text{ mode}). \quad (3.15b)$$

Values of $x = (\hbar\Omega/2\Delta)$ are plotted as a function of the left-hand sides of these equations in Fig. 6. The plot shows that when g_L becomes larger than g_0, the frequency Ω of the Γ_{LM} mode becomes imaginary, indicating that the system is unstable when described by a ground state formed with s-state pairing. Therefore, if g_L is the largest coupling constant present, the ground state should be formed from pair functions having L-type symmetry. The pair spin function is singlet or triplet depending on whether L is even or odd, since the wave function describing the exciton state must be antisymmetric on the interchange of all coordinates of the quasi-particle pair involved.

The growth of the Γ_{LM} modes for $g_L > g_0$ also indicates that the Λ_{LM} modes have no physical existence. As is seen in Fig. 6, a Λ_{LM} exciton cannot exist unless $g_L > g_0$. However, when such a coupling strength is reached, the corresponding Γ_{LM} exciton is unstable so that the system decays before the Λ_{LM} mode can come into existence. Figure 6 also indicates the 2L-fold M degeneracy of $q = 0$ L-state excitons.

It should be mentioned that a continuum of scattering state solutions is obtained from (3.14b) corresponding to the vanishing of the denominator of the integrand. One such state exists between two successive unperturbed levels, $E_k + E_{k+q}$. Although the energy of a scattering state solution is unaltered from its value in the absence of interactions, its wave function is strongly modified since each particle is surrounded by a depletion of the same type of particle leading to the backflow picture mentioned above.

(2) $\vec{q}$ Finite Case

From Eq. (3.10) it is seen that L is not strictly a good quantum number for the case of finite $\vec{q}$ since the coherence factors and $\nu_k(\vec{q})$ now have a polar angle dependence. Because of the complexity of this dependence, the sum Σ_k cannot be carried out exactly. We approximate

$$\mathcal{E}_{k+q} = \mathcal{E}_k + \hbar^2 \frac{\vec{k}\cdot\vec{q}}{m} + \frac{\hbar^2 q^2}{2m},$$

by

$$\mathcal{E}_{k+q} \simeq \mathcal{E}_k + \beta\mu \qquad (3.15c)$$

where $\beta = \hbar v_0 q$, $\mu = \cos\theta$, and v_0 is the velocity of a particle at the Fermi surface. This leads to an error of order $q/k_F \ll 1$. The integrals $I_{ab\ldots}$ are of the same form as those in the $q = 0$ case. To perform the angular integral, we expand the denominator of the integrand

$$I_{ab\ldots} = \frac{1}{(2\pi)^3} \int \frac{a(\vec{k},\vec{q})\,b(\vec{k},\vec{q})\ldots}{(\hbar\Omega)^2 - \gamma_k^2(\vec{q})^2}\, k^2 dk, \tag{3.16}$$

in powers of β. This procedure is valid so long as $\beta < \hbar\Omega - 2\Delta$. The integrals over k are then of the form

$$I_{ab\ldots} = I_{ab\ldots}{}^{0} + \mu I_{ab\ldots}{}^{1} + \mu^2 I_{ab\ldots}{}^{2} + \ldots , \tag{3.17}$$

with superscripts indicating the powers of β involved. Keeping terms through order β^2 and using the relations

$$\cos\theta = \mu = (4\pi/3)^{1/2}\, Y_{10}(\theta)$$

and

$$\cos^2\theta = \mu^2 = \tfrac{2}{3}(4\pi/5)^{1/2}\, Y_{20}(\theta) + (4\pi/9)^{1/2}\, Y_{00} ,$$

the equations for Λ and Γ (3.10) become

$$\Lambda_{LM}(\vec{q}) = V_L \int d\omega \Big\{ 2\left(\tfrac{4\pi}{3}\right)^{\frac{1}{2}} I'_{\nu mn}\, Z(\vec{q})\, Y^*_{LM} Y_{10} + \Big[\left((4\pi)^{\frac{1}{2}} I^0_{\nu m^2} + \left(\tfrac{4\pi}{9}\right)^{\frac{1}{2}} I^2_{\nu m^2}\right) Y_{00} + \tfrac{2}{3}\left(\tfrac{4\pi}{5}\right)^{\frac{1}{2}} I^2_{\nu m^2} Y_{20} \Big] \times \sum_{\ell} Y^*_{LM} Y_{\ell M} \Lambda_{\ell M}(\vec{q}) + \left(\tfrac{4\pi}{3}\right)^{\frac{1}{2}} I'_{\nu\Omega\ell n}\, Y_{10} \sum_{\ell} Y^*_{LM} Y_{\ell M}\, \Gamma_{\ell M}(\vec{q}) \Big\}, \tag{3.18a}$$

$$\Gamma_{LM}(\vec{q}) = V_L \int d\omega \Big\{ 2 \Big[\Big((4\pi)^{1/2} I^{0}_{k\Omega lm} + \Big(\frac{4\pi}{9}\Big)^{1/2} I^{2}_{k\Omega lm} \Big) Y_{00}$$

$$+ \frac{2}{3}\Big(\frac{4\pi}{5}\Big)^{1/2} I^{2}_{k\Omega lm}\, Y_{20} \Big] Z(\vec{q})\, Y^{*}_{LM} +$$

$$\Big(\frac{4\pi}{3}\Big)^{1/2} I'_{k\Omega ln}\, Y_{10} \sum_{l} Y^{*}_{LM} Y_{lM}\, \Lambda_{lM}(\vec{q}) + \Big[\Big((4\pi)^{1/2} I^{0}_{\gamma l^2}$$

$$+ \Big(\frac{4\pi}{9}\Big)^{1/2} I^{2}_{\gamma l^2} \Big) Y_{00} + \frac{2}{3}\Big(\frac{4\pi}{5}\Big)^{1/2} I^{0}_{\gamma l^2}\, Y_{20} \Big] \sum_{l} Y^{*}_{LM} Y_{lM}\, \Gamma_{lM}(\vec{q}) \Big\}. \tag{3.18b}$$

With the relation

$$\int d\omega\, Y^{*}_{l_3 m_3} Y_{l_2 m_2} Y_{l_1 m_1} = \Big[\frac{(2l_1+1)(2l_2+1)}{4\pi(2l_3+1)} \Big]^{1/2}$$

$$\times C(l_1 l_2 l_3; m_1 m_2 m_3)\, C(l_1 l_2 l_3; 000),$$

where the C's are usual Clebsch-Gordan coefficients,[11] Eqs. (3.18) become

$$\Lambda_{LM}(\vec{q}) = V_L \Big\{ 2\left(\tfrac{4\pi}{3}\right)^{\frac{1}{2}} I'_{\nu m n} Z(\vec{q})\, \delta_{L1}\, \delta_{M0} +$$

$$\left((4\pi)^{\frac{1}{2}} I^{0}_{\nu n^2} + \left(\tfrac{4\pi}{9}\right)^{\frac{1}{2}} I^{2}_{\nu n^2} \right) \sum_{\ell} \left[\frac{2\ell+1}{4\pi(2L+1)} \right]^{\frac{1}{2}}$$

$$\times C(0\ell L; 0MM)\, C(0\ell L; 000)\, \Lambda_{\ell M}(\vec{q}) +$$

$$\tfrac{2}{3}\left(\tfrac{4\pi}{5}\right)^{\frac{1}{2}} I^{2}_{\nu n^2} \sum_{\ell} \left[\frac{5(2\ell+1)}{4\pi(2L+1)} \right]^{\frac{1}{2}} C(2\ell L; 0MM)$$

$$\times C(2\ell L; 000)\, \Lambda_{\ell M}(\vec{q}) + \left(\tfrac{4\pi}{3}\right)^{\frac{1}{2}} I'_{n\Omega\ell n} \sum_{\ell} \left[\frac{3(2\ell+1)}{4\pi(2L+1)} \right]^{\frac{1}{2}}$$

$$\times C(1\ell L; 0MM)\, C(1\ell L; 000)\, \Gamma_{\ell M}(\vec{q}) \Big\},$$

(3.19a)

$$\Gamma_{LM}(\vec{q}) = V_L \Big\{ 2 \Big[\Big((4\pi)^{1/2} I^0_{n\Omega \ell m} + \left(\tfrac{4\pi}{9}\right)^{1/2} I^2_{n\Omega \ell m} \Big) \delta_{L0} \delta_{M0}$$

$$+ \tfrac{2}{3} \left(\tfrac{4\pi}{5}\right)^{1/2} I^2_{n\Omega \ell m} \delta_{L2} \delta_{M2} \Big] Z(\vec{q}) + \Big((4\pi)^{1/2} I^0_{\nu \ell^2} + \left(\tfrac{4\pi}{9}\right)^{1/2} I^2_{\nu \ell^2} \Big)$$

$$\times \sum_{\ell} \left[\tfrac{(2\ell+1)}{4\pi(2L+1)} \right]^{1/2} C(0\ell L; 0MM)\, C(0\ell L; 000)\, \Gamma_{\ell M}(\vec{q}) +$$

$$\tfrac{2}{3} \left(\tfrac{4\pi}{5}\right)^{1/2} I^2_{\nu \ell^2} \sum_{\ell} \left[\tfrac{5(2\ell+1)}{4\pi(2L+1)} \right]^{1/2} C(2\ell L; 0MM) \times$$

$$C(2\ell L; 000)\, \Gamma_{\ell M}(\vec{q}) + \left(\tfrac{4\pi}{3}\right)^{1/2} I'_{n\Omega \ell n} \sum_{\ell} \left[\tfrac{3(2\ell+1)}{4\pi(2L+1)} \right]^{1/2}$$

$$\times\, C(1\ell L; 0MM)\, C(1\ell L; 000)\, \Lambda_{\ell M}(\vec{q}) \Big\}.$$

(3.19b)

As in the $q \to 0$ case, the Coulomb field represented by the presence of the $Z(\vec{q})$ term does not couple into the equations of motion except for the longitudinal modes $M = 0$. Discussion of this case is deferred and the transverse cases $M \neq 0$ are now considered. For a given $M \neq 0$, Eqs. (3.19) represent a set of 2N linear simultaneous equations in Λ_{LM} and Γ_{LM}, where N is the number of terms present in the spherical harmonic decomposition of the two-body interaction (3.7). It follows that for a given set of V_L's the normal mode frequencies of the system may be obtained by setting the determinant of the coefficients of the Λ_{LM}'s and Γ_{LM}'s equal to zero. Once the frequencies have been obtained, the Λ_{LM}'s, Γ_{LM}'s, and the transformation coefficients f and g may be determined.

For simplicity we consider the case for which all but two of the V_L's vanish. It is assumed that the two-body potential consists of a term V_0, corresponding to the BCS parameter and another, V_L, representing the angular dependence of the interaction. Since M has been taken as nonzero, it is seen that the simplified V_0 and V_L potential allows the modes to be characterized by a quantum number L within the approximations of the calculation, due to Λ_{0M} and Γ_{0M} vanishing identically for $M \neq 0$. Thus, we may speak of a p-,d-,... state exciton when the additional term in the potential has L = 1,2,... type angular dependence.

If the potential contains s- and p-wave potentials,

$$V(\vec{R},\vec{R}') = V_0\, Y_{00}^*(\theta,\varphi)\, Y_{00}(\Theta,\Phi) + V_1\, Y_{1,\pm1}^*(\theta,\varphi)\, Y_{1,\pm1}(\Theta,\Phi), \tag{3.20}$$

the dispersion relations obtained from (3.19) are found to be

$$\frac{1}{V_1} = \left(I^0_{\nu n^2} + \frac{1}{5} I^2_{\nu n^2}\right), \quad [\Lambda_{1,\pm 1}(\vec{q}) \text{ modes}], \tag{3.21a}$$

$$\frac{1}{V_1} = \left(I^0_{\nu \ell^2} + \frac{1}{5} I^2_{\nu \ell^2}\right), \quad [\Gamma_{1,\pm 1}(\vec{q}) \text{ modes}]. \tag{3.21b}$$

We discard the Λ mode since it does not exist if the system is stable. The dispersion relation (3.21b) for the $\Gamma_{1,\pm 1}(\vec{q})$ mode, when rewritten in terms of explicit expressions for the integrals $I_{\nu \ell^2 0}$ and $I_{\nu \ell^2 2}$ becomes

$$(\xi_0 q)^2 = -\frac{10\chi^3}{\pi^2}\left[\frac{\frac{2}{g_1} - \frac{2}{g_0} - \frac{2\chi \arcsin\chi}{\sqrt{1-\chi^2}}}{\frac{\chi}{1-\chi^2} - \frac{\arcsin\chi}{\sqrt{1-\chi^2}} + \frac{\chi^2}{2}\frac{\arcsin\chi}{(1-\chi^2)^{3/2}}}\right] \tag{3.22}$$

where $x = \kappa\Omega_{1,\pm 1}\sqrt{2\Delta} < 1$. This dispersion relation is plotted in Fig. 7 for two values of g_1 with $g_0 = 0.25$. From the figure, it is seen that the curve intersects the origin for $g_1 = g_0$. For a value $g_1 < g_0$ there is a minimum value of $x = x_m$ given by $(1/g_1 - 1/g_0) = x_m \arcsin x_m/(1 - x_m^2)^{1/2}$, in agreement with the results of the last section for the $q \to 0$ case.

(3) The s-State Exciton

The above discussion was restricted to that of the transverse, $M \neq 0$, excitations in which the Coulomb interaction term $Z(\vec{q})$ did not enter into the equations for $\Lambda_{LM}(\vec{q})$ and $\Gamma_{LM}(\vec{q})$. Before discussing the $M = 0$ cases, it should be emphasized that the equations of motion (3.1) which are the basis of this paper are those linearized by Anderson about the BCS ground state based on s-state pairing of the electrons. As Anderson[3,12] has pointed out, it is the s-state exciton which corresponds to a plasmon excitation, due to $Z(\vec{q})$ coupling into the equations of motion.

The $L = 0$ mode is considered in the $q \to 0$ limit. Because of the singular nature of the direct interaction, it is not possible to set $q \equiv 0$ in the calculation, so that the limit $q \to 0$ must be taken. For our starting point, we consider Eq. (3.13b) for the $\Gamma_{00}(\vec{q})$ mode in the $q \to 0$ case:

$$(1 - V_0 I^0_{\nu\ell^2})\Gamma_{00} = \lim_{q\to 0} Z(\vec{q}) 2 V_0 I^0_{\hbar\Omega\ell m} . \tag{3.23}$$

From the definitions (3.5) and (3.9) an expression for $Z(q)$ is obtained:

$$Z(\vec{q}) = \frac{V_D(q)}{1 - 8\pi V_D(q)(I^0_{\nu\ell m^2} + I^2_{\nu\ell m^2}/3)} \times$$
$$\left\{\left[(4\pi)^{\frac{1}{2}} I^0_{\hbar\Omega\ell m} + \left(\frac{4\pi}{9}\right)^{\frac{1}{2}} I^2_{\hbar\Omega\ell m}\right]\Gamma_{00}(\vec{q}) + \right.$$
$$\left.\left(\frac{4\pi}{3}\right)^{\frac{1}{2}} I^0_{\ell m n} \Lambda_{10}(\vec{q}) + \frac{2}{3}\left(\frac{4\pi}{5}\right)^{\frac{1}{2}} I^2_{\hbar\Omega\ell m} \Gamma_{20}(\vec{q})\right\}. \tag{3.24}$$

Since the L = 0 mode excitation energy is being considered, only the $\Gamma_{00}(q)$ term in (3.24) need be used in substituting for Z(q) into (3.23). Rearrangement of terms then gives:

$$1 = \lim_{q\to 0} 8\pi V_D(q) \frac{1}{1 - V_0 I^0_{\mu\ell^2}} \times$$
$$\{ I^0_{\kappa\Omega\ell m} (I^0_{\kappa\Omega\ell m} + I^2_{\kappa\Omega\ell m}/3) V_0 +$$
$$(I^0_{\nu m^2} + I^2_{\nu m^2}/3)(1 - V_0 I^0_{\mu\ell^2}) \}. \qquad (3.25)$$

Since $V_D(q) \sim 1/q^2$, Eq. (3.26) indicates that in order for the limit to be finite, the terms in the numerator which are independent of q must vanish:

$$(I^0_{\kappa\Omega\ell m})^2 V_0 + I^0_{\nu m^2} (1 - V_0 I^0_{\mu\ell^2}) = 0. \qquad (3.26)$$

The validity of (3.26) is shown by considering the explicit form of the integrals involved:

$$I^{0}_{\nu m^2} = \int_{-\hbar\omega_c}^{\hbar\omega_c} \frac{2N(\xi)\,\Delta^2/E}{(\hbar\Omega)^2 - 4E^2}\, d\xi, \tag{3.27a}$$

$$\begin{aligned} I^{0}_{\hbar\Omega m} &= \hbar\Omega \int_{-\hbar\omega_c}^{\hbar\omega_c} \frac{N(\xi)\,\Delta/E}{(\hbar\Omega)^2 - 4E^2}\, d\xi \\ &= \hbar\Omega \int_{-\hbar\omega_c}^{\hbar\omega_c} N(\xi) \left\{ \frac{\Delta}{(\hbar\Omega)^2 E} + \right. \\ &\qquad \left. \frac{\Delta}{E}\left(\frac{1}{(\hbar\Omega)^2 - 4E^2} - \frac{1}{(\hbar\Omega)^2} \right) \right\} d\xi \\ &= \left\{ \frac{-2\Delta}{\hbar\Omega V_0} + \int_{\hbar\omega_c}^{\hbar\omega_c} \frac{N(\xi)\,4\Delta E}{\hbar\Omega\left[(\hbar\Omega)^2 - 4E^2\right]}\, d\xi \right. \\ &= \frac{2\Delta}{\hbar\Omega} \left\{ -\frac{1}{V_0} + I^{0}_{\nu\ell^2} \right\}, \end{aligned} \tag{3.27b}$$

where the BCS integral equation[1] for V_0 has been used to obtain the first term on the right. With the use of these relations, (3.26) becomes:

$$\frac{2\Delta}{\hbar\Omega} V_0 \left(-\frac{1}{V_0} + I_{\nu\ell^2}\right) \hbar\Omega \int_{-\hbar\omega_c}^{\hbar\omega_c} \frac{N(\xi)\Delta/E}{(\hbar\Omega)^2 - 4E^2}\, d\xi +$$

$$2\int_{-\hbar\omega_c}^{\hbar\omega_c} \frac{N(\xi)\Delta^2/E}{(\hbar\Omega)^2 - 4E^2} \left(1 - V_0 I_{\nu\ell^2}\right) d\xi = 0.$$

With the validity of (3.26) established, (3.25) reduces to

$$1 = \lim_{q\to 0} \frac{8\pi V_D(q)}{3} \frac{1}{1 - V_0 I^0_{\nu\ell^2}} \times$$

$$\left\{ V_0 I^0_{\hbar\Omega\ell m}\, I^2_{\hbar\Omega\ell m} + I^2_{\nu\ell m} \left(1 - V_0 I^0_{\nu\ell^2}\right) \right\}. \tag{3.28}$$

To determine the existence of a plasma oscillation for the L = 0 mode, (3.28) must have a solution for $x = (\hbar\Omega/2\Delta) \gg 1$. Under this condition the term $V_0 I_{\nu\ell 2}{}^0$ in the denominator is much less than unity and may be dropped. The integrals involved in (3.28) are evaluated for $x \gg 1$ so that, to order $1/x^2$, (3.28) reduces to

$$1 = \frac{\pi^2}{6x^2} V_D(q)\, q^2 N(0)\, \xi_0^2. \tag{3.29}$$

Using $V_D(q) = 4\pi e^2/q^2$ and $e^2\xi_0{}^2 N(0) = (3/2\pi^3)(\hbar\omega_p/2\Delta)^2$, where $\omega_p{}^2 = 4\pi ne^2/m$, (3.29) gives $\Omega = \omega_p$ so that the excitation frequency of this mode is the plasma frequency.

(4) The L = 1, M = 0 Mode

To complete the investigation of the collective states present when only the V_0 and V_1 terms are kept in the potential expansion (3.7), we must determine the dispersion relation for the $\Gamma_{10}(q)$ mode. Setting M = 0 in (3.19b) we obtain two simultaneous equations involving $\Gamma_{00}(\vec{q})$ and $\Gamma_{10}(\vec{q})$. There is no mixing of these modes in the equations. The Γ_{00} dispersion relation gives the plasma frequency as discussed above while the $\Gamma_{10}(\vec{q})$ mode dispersion relation becomes

$$\frac{1}{V_1} = \left(I^0_{\nu\ell^2} + \frac{3}{5} I^2_{\nu\ell^2}\right). \tag{3.30}$$

In Sec. III (2) we found the dispersion relation for the $\Gamma_{1\pm1}(q)$ modes to be

$$\frac{1}{V_1} = \left(I^0_{\nu\ell^2} + \frac{1}{5} I^2_{\nu\ell^2}\right). \tag{3.21b}$$

Thus the $\Gamma_{10}(\vec{q})$ dispersion relation can be obtained by letting $\vec{q} \to \vec{q}\sqrt{3}$ in (3.22), indicating that for a given wave vector $\vec{q}$ the excitation energy of the longitudinal $\Gamma_{10}(q)$ mode is raised above that of the transverse $\Gamma_{1\pm1}(\vec{q})$ modes.

IV. CORRECTIONS TO THE ANDERSON-RICKAYZEN EQUATIONS

We consider here the terms in the linearized equations neglected by Anderson and Rickayzen. For simplicity we treat these terms only in the $\vec{q} \to 0$ case. In the equation for $b_k^\dagger(\vec{q})$, the terms shown in Figs. 2(h) and 2(i) were neglected. They contribute the factor

$$-u_k v_k \sum_{k'} V(\vec{k},\vec{k}')(\Delta_{k'}/2E_{k'}) \times (\gamma^{+}_{k'0}\gamma^{+}_{k'1} + \gamma_{k'1}\gamma_{k'0}) \tag{4.1}$$

to the right hand side of (3.1a) in the limit $\vec{q} \to 0$, while the negative of this factor is added to the right-hand side of (3.1b). The exchange scattering vertex shown in Fig. 3(a) was neglected in the equation for $\rho_{k\sigma}(\vec{q})$. Its contribution,

$$\sum_{k'} C^{+}_{k'+q\sigma} C_{k'\sigma} [V(\vec{k}',k) v^2_{k+q} - V(\vec{k}+\vec{q},\vec{k}'+\vec{q}) v_k^2], \tag{4.2}$$

vanishes as $\vec{q} \to 0$ and does not affect the energy of the exciton states in this limit. The inclusion of (4.1) adds the term

$$-\frac{1}{2}\frac{\Delta_k}{E_k}\sum_{k'} V(\vec{k},\vec{k}')\frac{\Delta_{k'}}{E_{k'}}(f_{k'}+g_{k'}) \tag{4.3}$$

to the right-hand side of (3.6a) and the negative of this term to the right-hand side of (3.6b). Introducing the variable

$$R_k = \sum_{k'} \frac{\Delta_k}{E_k} V(\vec{k},\vec{k}')(f_{k'}+g_{k'}), \tag{4.4}$$

one finds the $M \neq 0$ exciton states satisfy the set of coupled equations:

$$\Gamma_{LM} = \Gamma_{LM} V_L \sum_k \frac{2E_k}{(\hbar\Omega)^2 - 4E_k^2} - R_{LM} V_L \sum_k \frac{\Delta_k}{E_k} \frac{\hbar\Omega}{(\hbar\Omega)^2 - 4E_k^2},$$

$$R_{LM} = \Gamma_{LM} V_L \sum_k \frac{\Delta_k}{E_k} \frac{\hbar\Omega}{(\hbar\Omega)^2 - 4E_k^2} - R_{LM} V_L \sum_k \frac{2\Delta_k^2}{E_k[(\hbar\Omega)^2 - 4E_k^2]}. \tag{4.5}$$

Setting the determinant of the coefficients equal to zero, one finds the dispersion relation

$$\left(\frac{1}{V_L} + I^0_{lm^2}\right)\left(\frac{1}{V_L} - I^0_{l^2}\right) + \left(I^0_{\hbar\Omega,lm}\right)^2 = 0, \tag{4.6}$$

or

$$\left(\frac{1}{g_L} + \frac{\text{Arcsin}\,x}{x(1-x^2)^{1/2}}\right)\left(\frac{1}{g_L} - \frac{1}{g_0} - \frac{x\,\text{Arcsin}\,x}{(1-x^2)^{1/2}}\right) + \frac{(\text{Arcsin}\,x)^2}{1-x^2} = 0, \tag{4.7}$$

for the energy of the Γ_{LM} exciton. The modification of the $\vec{q} \to 0$ exciton energy given by (4.7) is shown in Fig. 6 for $g_0 = 0.25$ and is seen to be small. A new type of excitation follows from (4.7) for $g_L < 0$, that is, a repulsive rather than attractive L-wave interaction between electrons. The energy of this state is shown in Fig. 8 as a function of $-g_L$ for $g_0 = 0.25$. From the form of the coherence factors entering the dispersion relation it appears the new state should be interpreted as a bound electron-hole pair in close analogy with the exciton states occurring in insulators. This interpretation is consistent with the fact that the electron-hole interaction is attractive when the corresponding electron-electron interaction is repulsive. Thus the electron-hole exciton arises solely from the terms neglected in the Anderson-Rickayzen equations.

In Sec. V we consider the role played by this type of exciton in the absorption of electromagnetic waves of frequencies lying within the gap.

V. ELECTROMAGNETIC ABSORPTION

Ginsberg, Richards and Tinkham[7] have measured the absorption of infrared radiation in bulk samples and the transmission through thin films of several superconductors. In samples of lead and tin they have found a precursor absorption existing for frequencies below that corresponding to the energy gap. The Rickayzen form of the Anderson equations of motion has been applied by Tsuneto[6] to the problem of the surface impedance at finite frequency. While he finds the existence of a precursor absorption due to the creation

of excitons, his results predict a value for the absorption which is an order of magnitude smaller than the experimental value. The Tsuneto analysis does not include the corrections to the Anderson-Rickayzen equations described in IV which give rise to the hole-particle excitons. In this section we calculate the infrared absorption due to hole-particle excitons in order to investigate how this absorption may modify the Tsuneto results and to see if the experimental results can be explained.

In order to calculate the absorption we must extend the equations (3.1) for $\gamma^{+}_{k+q_0}\gamma_{k_1}$ and $\gamma_{k+q_1}\gamma_{k_0}$ to treat particle-hole excitons with a finite center of mass momentum $\hbar\vec{q}$. Once again the quasi-particle conserving operators $\gamma^{+}_{k+q\sigma}\gamma_{k\sigma}$ are not considered since they do not contribute to the dispersion relation for the excitons nor to their absorption.

We begin by defining the quantity $D(\vec{k},\vec{k}')$ by

$$D(k,k') = u_{k'}v_k(u_k^2+v_{k'}^2). \tag{5.1}$$

With this definition, the non quasi-particle conserving contribution to the equation of motion (3.1a) for $\gamma^{+}_{k+q_0}\gamma_{k_1}$ due to the inclusion of the terms shown in Figs. 2(h) and 2(i) is

$$C_k(\vec{q}) = -\sum_{k'} \frac{1}{2}\left(V(\vec{k},\vec{k}')+V(\vec{k}+\vec{q},\vec{k}'+\vec{q})\right)\times$$

$$\left\{\left[D(\vec{k},\vec{k}+\vec{q})\,u_{k'+q}\,v_{k'} + D(\vec{k}+\vec{q},\vec{k})\,u_{k'}\,v_{k'+q}\right]\gamma^{+}_{k'+q0}\gamma^{+}_{k'1}\right.$$

$$\left. + \left[D(\vec{k},\vec{k}+\vec{q})\,v_{k'+q}\,u_{k'} + D(\vec{k}+\vec{q},\vec{k})\,u_{k'+q}\,v_{k'}\right]\gamma_{k'+q1}\gamma_{k'0}\right\}. \qquad (5.2a)$$

The contribution to equation (3.1b) is

$$-\bar{C}_k(\vec{q}) = \sum_{k'} \frac{1}{2}\left(V(\vec{k},\vec{k}')+V(\vec{k}'+\vec{q},\vec{k}+\vec{q})\right)\times$$

$$\left\{\left[D(\vec{k}+\vec{q},\vec{k})\,u_{k'}\,v_{k'+q} + D(\vec{k},\vec{k}+\vec{q})\,u_{k'+q}\,v_{k'}\right]\gamma_{k'+q1}\gamma_{k'0}\right.$$

$$\left. + \left[D(\vec{k}+\vec{q},\vec{k})\,v_{k'}\,u_{k'+q} + D(\vec{k},\vec{k}+\vec{q})\,u_{k'}\,v_{k'+q}\right]\gamma^{+}_{k'+q0}\gamma^{+}_{k'1}\right\}. \qquad (5.2b)$$

In order to investigate the electromagnetic absorption due to the creation of excitons we consider the effect on the equations of motion (3.1a) and (3.1b) of an externally applied transverse vector potential

$$\vec{A}(\vec{R},t) = \vec{a}(\vec{q}) \exp\left[i\vec{q}\cdot\vec{R} - i(\omega+i\eta)t\right]. \tag{5.3}$$

With $\alpha = (e\mathcal{A}/2mc)$, the inclusion of the vector potential and the corrections (5.2), the equations (3.1a) and (3.1b) are rewritten as

$$\begin{aligned}\left[H, \gamma^{\dagger}_{k+q0}\gamma^{\dagger}_{k1}\right] &= V_k(\vec{q})\gamma^{\dagger}_{k+q0}\gamma^{\dagger}_{k1} + V_D(\vec{q})\rho(\vec{q})m(\vec{R},\vec{q}) \\ &+ \tfrac{1}{2}n(\vec{R},\vec{q})B_k(\vec{q}) - \tfrac{1}{2}\ell(\vec{R},\vec{q})A_k(\vec{q}) + C_k(\vec{q}) \\ &+ \alpha\rho(\vec{R},\vec{q})\,\vec{a}(\vec{q})\cdot(2\vec{k}+\vec{q})\exp\left[-i(\omega+i\eta)t\right],\end{aligned} \tag{5.4a}$$

and

$$\begin{aligned}\left[H, \gamma_{k+q1}\gamma_{k0}\right] &= -V_k(\vec{q})\gamma_{k+q1}\gamma_{k0} - V_D(\vec{q})\rho(\vec{q})m(\vec{R},\vec{q}) \\ &- \tfrac{1}{2}n(\vec{R},\vec{q})B_k(\vec{q}) - \tfrac{1}{2}\ell(\vec{R},\vec{q})A_k(\vec{q}) - \overline{C}_k(\vec{q}) \\ &+ \alpha\rho(\vec{R},\vec{q})\,\vec{a}(\vec{q})\cdot(2\vec{k}+\vec{q})\exp\left[-i(\omega+i\eta)t\right].\end{aligned} \tag{5.4b}$$

The variables $A_k(\vec{q})$, $B_k(\vec{q})$, and $\rho_k(\vec{q})$ have been defined previously by equations (2.18). It is the three coupled integral equations for these variables which are the basis for Tsuneto's analysis of the electromagnetic absorption. With the inclusion of correction terms due to the presence of the hole-particle excitons, the two new variables, $C_k(\vec{q})$ and $\bar{C}_k(\vec{q})$, of equations (5.2) are coupled into the equations for the three original variables. Tsuneto has shown that the cross terms involving $\rho_k(\vec{q})$ and $B_k(\vec{q})$ vanish so that his absorption is determined by the one collective variable $A_k(\vec{q})$. In our analysis, we have three coupled integral equations in $A_k(\vec{q})$, $C_k(\vec{q})$ and $\bar{C}_k(\vec{q})$.

It becomes convenient to introduce a change of variables with the new set of definitions:

$$T_{1k}(\vec{q}) = \sum_{k'} V(k,k') \times \left\{ u_{k'+q} v_{k'} \gamma^{+}_{k+q0} \gamma^{+}_{k'1} + v_{k'+q} u_{k'} \gamma_{k'+q1} \gamma_{k'0} \right\}, \tag{5.5a}$$

$$T_{2k}(\vec{q}) = \sum_{k'} V(k,k') \times \left\{ u_{k'} v_{k'+q} \gamma^{+}_{k'+q0} \gamma^{+}_{k'1} + u_{k'+q} v_{k'} \gamma_{k'+q1} \gamma_{k'0} \right\}, \tag{5.5b}$$

$$T^{\pm}_{k}(\vec{q}) = T_{1k}(\vec{q}) \pm T_{2k}(\vec{q}). \tag{5.5c}$$

We also note that $D(\vec{k},\vec{k}+\vec{q}) + D(\vec{k}+\vec{q},\vec{q}) = \ell^2(\vec{k},\vec{q})\, m(\vec{k},\vec{q})$, and $D(\vec{k},\vec{k}+\vec{q}) - D(\vec{k}+\vec{q},\vec{q}) = -n^2(\vec{k},\vec{q})\, P(\vec{k},\vec{q})$. We will work with the following three coupled integral equations in $A_k(\vec{q})$ and $T_k^{\pm}(\vec{q})$:

$$A_k(\vec{q}) = \sum_{k'} V(\vec{k},\vec{k}') \frac{1}{(\hbar\omega+i\eta)^2 - v_{k'}^2(\vec{q})} \times$$

$$\{\ell^2(\vec{k}',\vec{q})\, v_{k'}(\vec{q})\, A_{k'}(\vec{q}) - 4\alpha\, p(\vec{k}',\vec{q})\, \ell(\vec{k}',\vec{q})\, v_{k'}(\vec{q})\, \vec{\alpha}(\vec{q})\cdot\vec{k}'$$

$$-\hbar\omega\, \ell^2(\vec{k}',\vec{q})\, m(\vec{k}',\vec{q})\, T_{k'}^{+}(\vec{q}) - \ell(\vec{k}',\vec{q})\, n^2(\vec{k}',\vec{q})\, p(\vec{k}',\vec{q})\, v_{k'}(\vec{q})\, T_{k'}^{-}(\vec{q})\}, \tag{5.6a}$$

$$T_k^{+}(\vec{q}) = \sum_{k'} V(\vec{k},\vec{k}') \frac{1}{(\hbar\omega+i\eta)^2 - v_{k'}^2(\vec{q})^2} \times$$

$$\{\hbar\omega\, \ell(\vec{k}',\vec{q})\, m(\vec{k}',\vec{q})\, A_{k'}(\vec{q}) - 4\alpha\hbar\omega\, m(\vec{k}',\vec{q})\, p(\vec{k}',\vec{q})\, \vec{\alpha}(\vec{q})\cdot\vec{k}'$$

$$-\ell^2(\vec{k}',\vec{q})\, m^2(\vec{k}',\vec{q})\, v_{k'}(\vec{q})\, T_{k'}^{+}(\vec{q}) - \hbar\omega\, m(\vec{k}',\vec{q})\, n^2(\vec{k}',\vec{q})\, p(\vec{k}',\vec{q})\, T_{k'}^{-}(\vec{q})\}, \tag{5.6b}$$

$$T_k^{-}(\vec{q}) = \sum_{k'} V(\vec{k},\vec{k}') \frac{1}{(\hbar\omega+i\eta)^2 - v_{k'}(\vec{q})^2} \times$$

$$\{v_{k'}(\vec{q})\, \ell(\vec{k}',\vec{q})\, p(\vec{k}',\vec{q})\, A_{k'}(\vec{q}) - 4\alpha\, v_{k'}(\vec{q})\, p^2(\vec{k}',\vec{q})\, \vec{\alpha}(\vec{q})\cdot\vec{k}'$$

$$-\hbar\omega\, \ell^2(\vec{k}',\vec{q})\, m(\vec{k}',\vec{q})\, p(\vec{k}',\vec{q})\, T_{k'}(\vec{q}) - v_{k'}(\vec{q})\, n^2(\vec{k}',\vec{q})\, p^2(\vec{k}',\vec{q})\, T_{k'}^{-}(\vec{q})\}. \tag{5.6c}$$

We again use the decomposition of the two body potential $V(\vec{k}',\vec{k})$ into spherical harmonics as described by equation (3.7). This allows us to define the three quantities $A_{\ell m}(\vec{q})$, $T_{1\ell m}(\vec{q})$ and $T_{2\ell m}(\vec{q})$:

$$A_{\ell m}(\vec{q}) = -V_\ell \sum_{k'} Y^*_{\ell m}(\theta,\varphi)\, \ell(k',\vec{q})\, (\gamma^+_{k'+q0}\gamma^+_{k'1} - \gamma_{k'+q1}\gamma_{k'0}), \tag{5.7a}$$

$$T_{1\ell m}(\vec{q}) = \sum_{k'} V_\ell\, Y^*_{\ell m}(\theta,\varphi)\, (u_{k'+q}\, v_{k'}\, \gamma^+_{k'+q0}\gamma^+_{k'1} + v_{k'+q}\, u_{k'}\, \gamma_{k'+q1}\gamma_{k'0}), \tag{5.7b}$$

$$T_{2\ell m}(\vec{q}) = \sum_{k'} V_\ell\, Y^*_{\ell m}(\theta,\varphi)\, (u_{k'}\, v_{k'+q}\, \gamma^+_{k'+q0}\gamma^+_{k'1} + u_{k'+q}\, v_{k'}\, \gamma_{k'+q1}\gamma_{k'0}). \tag{5.7c}$$

Analogous to equation (5.5c) we set

$$T^{\pm}_{\ell m}(\vec{q}) = T_{1\ell m}(\vec{q}) \pm T_{2\ell m}(\vec{q}) . \tag{5.7d}$$

With these definitions and use of the orthonormality properties of the spherical harmonics, equations (5.6) may be reduced to

$$A_{LM}(\vec{q}) = V_L \sum_{k'} \frac{1}{(\hbar\omega + i\eta)^2 - v_{k'}^2(\vec{q})^2} \times$$

$$\{ \ell^2(k';\vec{q})\, v_{k'}(\vec{q}) \sum_{\ell} Y_{LM}^{*}\, Y_{\ell M}\, A_{\ell M}(\vec{q})$$

$$- \hbar\omega\, \ell^3(k';\vec{q})\, m(k';\vec{q}) \sum_{\ell} Y_{LM}^{*}\, Y_{\ell M}\, T_{\ell M}^{+}(\vec{q})$$

$$- \ell(k';\vec{q})\, n^2(k';\vec{q})\, p(k';\vec{q})\, v_{k'}(\vec{q}) \sum_{\ell} Y_{LM}^{*}\, Y_{\ell M}\, T_{\ell M}^{-}(\vec{q})$$

$$- 4\alpha\, p(k';\vec{q})\, \ell(k';\vec{q})\, v_{k'}(\vec{q})\, \vec{a}(\vec{q}) \cdot \hat{k}'\, Y_{LM}^{*}\, (\delta_{M,1} + \delta_{M,-1}) \},$$

(5.8a)

$$T_{LM}^{+}(\vec{q}) = V_L \sum_{k'} \frac{1}{(\hbar\omega + i\eta)^2 - v_{k'}^2(\vec{q})} \times$$

$$\{ \hbar\omega\, \ell(k';\vec{q})\, m(k';\vec{q}) \sum_{\ell} Y_{LM}^{*}\, Y_{\ell M}\, A_{\ell M}(\vec{q})$$

$$- \ell^2(k';\vec{q})\, m^2(k';\vec{q})\, v_{k'}(\vec{q}) \sum_{\ell} Y_{LM}^{*}\, Y_{\ell M}\, T_{\ell M}^{+}(\vec{q})$$

$$- \hbar\omega\, m(k';\vec{q})\, n^2(k';\vec{q})\, p(k';\vec{q}) \sum_{\ell} Y_{LM}^{*}\, Y_{\ell M}\, T_{\ell M}^{-}(\vec{q})$$

$$- 4\alpha\, \hbar\omega\, m(k';\vec{q})\, p(k';\vec{q})\, \vec{a}(\vec{q}) \cdot \hat{k}'\, Y_{LM}^{*}\, (\delta_{M,1} + \delta_{M,-1}) \},$$

(5.8b)

$$T_{LM}^{-}(\vec{q}) = V_L \sum_{k'} \frac{1}{(\hbar\omega+i\eta)^2 - v_{k'}(\vec{q})^2} \times$$

$$\{ v_{k'}(\vec{q})\, \ell(\vec{k}';\vec{q})\, p(\vec{k}';\vec{q}) \sum_{\ell} Y_{LM}^{*} Y_{\ell M} A_{\ell M}(\vec{q})$$

$$-\hbar\omega\, \ell^2(\vec{k}';\vec{q})\, m(\vec{k}';\vec{q})\, p(\vec{k}';\vec{q}) \sum_{\ell} Y_{LM}^{*} Y_{\ell M} T_{\ell M}^{+}(\vec{q})$$

$$-\hbar^2(\vec{k}';\vec{q})\, p^2(\vec{k}';\vec{q})\, v_{k'}(\vec{q}) \sum_{\ell} Y_{LM}^{*} Y_{\ell M} T_{\ell M}^{-}(\vec{q})$$

$$-4\alpha\, v_{k'}(\vec{q})\, p^2(\vec{k}';\vec{q})\, \vec{a}(\vec{q})\cdot\hat{k}'\, Y_{LM}^{*}(\delta_{M,1}+\delta_{M,-1}) \}. \qquad (5.8c)$$

We must calculate the correction to the BCS paramagnetic current due to the transverse collective excitations which are included in equations (5.8). Then, following the method of Tsuneto, we determine the surface resistance of a bulk superconductor for frequencies less than the gap.

The paramagnetic part of the current density is

$$\vec{J}_p(\vec{q}) = -c\alpha \sum_{k} (2\vec{k}+\vec{q})\, p(\vec{k},\vec{q}) \times$$
$$(\gamma_{k+q,0}^{+}\gamma_{k,1}^{+} - \gamma_{k+q,1}\gamma_{k0}). \qquad (5.9)$$

By the methods described in section III, we take matrix elements of equations (5.4) between the renormalized ground state $|0\rangle$ and the state $|1(\vec{q},\alpha)\rangle$ containing one elementary excitation. The results of this, together with our expressions for $A_k(\vec{q})$ and $T_k^{\pm}(\vec{q})$, allow the paramagnetic current density to be rewritten as

$$\vec{J}_p(\vec{q},\omega) = -c\alpha \sum_k (2\vec{k}+\vec{q}) \frac{P(k,\vec{q})}{(\hbar\omega+i\eta)^2 - V_k^2(\vec{q})} \times$$
$$\{ -\ell(k,\vec{q})\, V_k(\vec{q})\, A_k(\vec{q})$$
$$-\hbar\omega\, \ell^2(k,\vec{q})\, m(k,\vec{q})\, T_k^{+}(\vec{q})$$
$$-V_k(\vec{q})\, n^2(k,\vec{q})\, p(k,\vec{q})\, T_k^{-}(\vec{q})$$
$$+ 2\alpha\, p(k,\vec{q})\, V_k(\vec{q})\, \vec{a}(\vec{q}) \cdot (2\vec{k}+\vec{q}) \}. \qquad (5.10)$$

Since the odd ℓ values in our potential expansion (3.7) cannot couple with a transverse field, we consider a simplified case in which the only terms present in the potential are V_0 and V_2. Furthermore, the presence of the Kronecker deltas (due to the transverse nature of the applied field) in the driving terms of equations

(5.8) indicate that we need only consider the $M = \pm 1$ cases in solving these equations. In addition, it is easily shown that $A_{21}(\vec{q}) = - A_{2,-1}(\vec{q})$ and $T^{\pm}_{21}(\vec{q}) = - T^{\pm}_{2,-1}(\vec{q})$. Therefore, we must solve equations (5.8) for the three collective coordinates $A_{21}(\vec{q})$ and $T^{\pm}_{21}(\vec{q})$. We reduce our notation by referring to these as $A(\vec{q})$ and $T^{\pm}(\vec{q})$.

We introduce the sums

$$S_{1ab\ldots} = V_2 \sum_{k'} \frac{Y_{21}^*(\theta,\varphi)\, Y_{21}(\theta,\varphi)\, ab\ldots}{(\hbar\omega)^2 - \nu_{k'}(\vec{q})^2} \tag{5.11}$$

and

$$S_{2ab\ldots} = V_2 \sum_{k'} \frac{Y_{21}^*(\theta,\varphi)\, ab\ldots}{(\hbar\omega)^2 - \nu_{k'}(\vec{q})^2}\,, \tag{5.12}$$

where each of the quantities a,b,c,... is one of the coherence factors, $\nu_k(\vec{q})$, $\hbar\omega$ or $\vec{a}(\vec{q})\cdot\vec{k}'$. With this notation, the three equations (5.8) for $A(\vec{q})$ and $T^{\pm}(\vec{q})$ may be rewritten

$$(1 - S_{1\ell^2\nu})A + S_{1\hbar\omega\ell^3 m} T^+ + S_{1\ell p n^2 \nu} T^- = -4\alpha\, S_{2p\ell\nu}\, \vec{a}\cdot\vec{k}', \tag{5.13a}$$

$$-S_{1\hbar\omega\ell m} A + (1 + S_{1\ell^2 m^2 \nu}) T^+ + S_{1\hbar\omega m n^2 p} T^- = -4\alpha\, S_{2\hbar\omega p}\, \vec{a}\cdot\vec{k}' m, \tag{5.13b}$$

$$-S_{1\ell p\nu} A + S_{1\hbar\omega\ell^2 mp} T^+ + (1 + S_{1n^2p^2\nu}) T^- = -4\alpha\, S_{2p^2\nu}\, \vec{a}\cdot\vec{k}'. \tag{5.13c}$$

When the angular integrations are considered, the orthonormality properties of the spherical harmonics reduce equation (5.13c) to

$$(1 + S_{1h^2p^2\nu})T^- = 0. \tag{5.14}$$

Therefore, a consistent solution to equations (5.13) may be obtained with the setting of $T^- \equiv 0$, since the quantity $(1 + S_{1h^2p^2\nu})$ is non-vanishing when evaluated at points of the dispersion relation determined by A and T^+ alone. The two equations to be solved are now

$$(1-S_{1\ell^2\nu})A + S_{1\kappa\omega\ell^3m}T = -4\alpha S_{2p\ell\nu}\vec{a}\cdot\vec{k}', \tag{5.15a}$$

$$-S_{1\kappa\omega\ell m}A + (1+S_{1\ell^2m^2\nu})T = -4\alpha S_{2\kappa\omega mp}\vec{a}\cdot\vec{k}', \tag{5.15b}$$

where we have written $T^+ = T$ for convenience of notation. The solutions are

$$A = P^{-1}\left[-4\alpha S_{2p\ell\nu}\vec{a}\cdot\vec{k}'(1+S_{1\ell^2m^2\nu}) + 4\alpha S_{2\kappa\omega mp}\vec{a}\cdot\vec{k}' S_{1\kappa\omega\ell^3m}\right], \tag{5.16a}$$

$$T = -P^{-1}\left[4\alpha S_{2\kappa\omega mp}\vec{a}\cdot\vec{k}'(1-S_{1\ell^2\nu}) + 4\alpha S_{2p\ell\nu}\vec{a}\cdot\vec{k}' S_{1\kappa\omega\ell m}\right], \tag{5.16b}$$

where

$$P=(1-S_{1\ell 2\ell})(1+S_{1\ell' m' 2\ell})+S_{1\kappa\omega\ell' m}\,S_{1\kappa\omega\ell m}. \tag{5.16c}$$

We now consider the calculation of the surface resistance. In equation (5.10) for the paramagnetic current density the first term gives the BCS paramagnetic current. The collective part may be written as

$$\vec{J}_{coll}(\vec{q},\omega)=c\alpha\sum_{k}\frac{4k_x\,p(\vec{k},\vec{q})}{(\hbar\omega+i\eta)^2-\nu_k^2(\vec{q})}\,Re\left(Y_{21}(\Theta,\Phi)\right) \tag{5.17}$$
$$\times\left[\ell(\vec{k},\vec{q})\,\nu_k(\vec{q})A+\hbar\omega\,\ell^2(\vec{k},\vec{q})\,m(\vec{k},\vec{q})\,T\right]\begin{pmatrix}\vec{a}\\ a\end{pmatrix}.$$

The kernel $K_{coll}(\vec{q},\omega)$ is defined by

$$\vec{J}_{coll}(\vec{q},\omega)=-\frac{c}{4\pi}\,K_{coll}(\vec{q},\omega)\,\vec{\alpha}(\vec{q},\omega), \tag{5.18}$$

and can be easily shown to be given by

$$K_{coll}(\vec{q},\omega)=\frac{8\pi\alpha}{a}\sum_{k}\sqrt{\frac{8\pi}{3}}\,k_F\,\frac{Y_{11}^{*}(\Theta,\Phi)\,Y_{21}(\Theta,\Phi)}{(\hbar\omega+i\eta)^2-\nu_k^2(\vec{q})}\,X$$
$$p(\vec{k},\vec{q})\left[\ell(\vec{k},\vec{q})\,\nu_k(\vec{q})A+\hbar\omega\,\ell^2(\vec{k},\vec{q})\,m(\vec{k},\vec{q})\,T\right]. \tag{5.19}$$

In this expression the quantity k_F appears as an approximation to $|\vec{k}|$ in k_x since the two-body potential is assumed to vanish except for $|\vec{k}|$ and $|\vec{k}'|$ near k_F.

To evaluate the surface resistance we assume the condition of specular reflection of the quasi-particles at the boundary of the superconductor. We should not expect our results to be too sensitive to the boundary conditions. For example, the penetration depth computed under the specular reflection condition differs little from that calculated in the random scattering limit. Also, the condition of random scattering involves mathematical complexities beyond the scope of our calculation.

With the specular reflection boundary condition the instantaneous field inside a bulk superconductor is given by[13]

$$a(\vec{q},\omega) = \frac{H(0)}{\pi} \frac{1}{q^2 - (\frac{\omega}{c})^2 + K(\vec{q},\omega)} \tag{5.20}$$

where H(0) is the magnetic field at the surface. The quantity ω^2/c^2 is neglected since we are interested in the infrared region where the wavelength of the electromagnetic wave is large compared to the penetration depth of the sample. For our rough calculation, we also neglect K_{coll} in the denominator, so that $K \simeq K_0$, the ordinary kernel. We further approximate K_0 by $1/\lambda_L^2$ which is valid for $\xi_0 q < 1$. The quantity λ_L is the London penetration depth.

The rate of absorption of a wave of frequency ω is

$$\begin{aligned} &2(2\pi)^2 \int_0^\infty Re\left[\vec{j}(\vec{q},\omega) \cdot \vec{E}(\vec{q},\omega)\right] dq \\ &= 2\pi(\hbar\omega) \int_0^\infty Im\left[K_{coll}(\vec{q},\omega)\, \vec{a}^2(\vec{q},\omega)\right] dq. \end{aligned} \tag{5.21}$$

With the substitution of our approximation for $\vec{a}(q,\omega)$ into (5.21), we get

$$R_s(\omega) = \frac{2(\hbar\omega)}{\pi} \lambda_L^4 \int_0^\infty Im\ K_{coll}(\vec{q},\omega)\, dq. \tag{5.22}$$

Thus we are interested in evaluating the imaginary part of $K_{coll}(\vec{q},\omega)$ given by equation (5.19). In section III(2) explicit expressions could not be obtained for the integrals involved in $\Lambda_{LM}(\vec{q})$ and $\Gamma_{LM}(\vec{q})$, and approximations were used to second order in the momentum transfer $\hbar\vec{q}$. Similar approximations are made in this section for the evaluation of A,T and subsequently Im $K_{coll}(\vec{q},\omega)$. With these approximations, the coupling constants g_L defined by (3.8), and $x = (\hbar\omega/2\Delta)$ we can obtain

$$Im\, K(\vec{q},\omega) = \frac{-\pi^2}{20} g_2 \left(\frac{\xi_0 q}{\lambda_L}\right)^2 \frac{1}{x^4} \frac{1}{1-x^2} \times$$
$$\frac{\delta(\vec{q}-\vec{q}(\omega))}{\left|\frac{\partial P}{\partial q}\right|_{q=q(\omega)}} \left\{ \left(-1+\frac{g_2}{g_0}\right) x^2(1-x^2) + \right.$$
$$\left[2\left(1-\frac{g_2}{g_0}\right)x + g_2 x^3\right](1-x^2)^{1/2} \operatorname{Arcsin} x$$
$$\left. - \left[\left(1-\frac{g_2}{g_0}\right) + x^2\right](\operatorname{Arcsin} x)^2 \right\}. \tag{5.23}$$

The quantity $q(\omega)$ is the solution to the equation $P(q,\omega) = 0$ which gives the dispersion relation for the exciton spectrum. The presence of the delta function term indicates the absorption by the system of a photon of frequency ω resulting in the creation of an exciton of energy $\hbar\omega$ and center of mass momentum $\hbar\vec{q}(\omega)$. Because of the delta function, the integral in expression (5.22) for $R_S(\omega)$ is easy to evaluate. We are interested in obtaining the ratio of $R_S(\omega)$ to the resistance of a normal metal. In the extreme non-classical (anomalous) limit, and with the boundary condition of specular reflection, the surface resistance of a normal metal is given by[14]

$$R_\infty(\omega) = (3)^{\frac{1}{6}}(\hbar\omega\lambda_L/4\Delta)^{\frac{2}{3}}\left(\frac{\ell}{\xi_0}\right)^{\frac{1}{3}}\Delta. \qquad (5.24)$$

From the data of Ginsberg, Richards and Tinkham the absorption in the gap has a maximum at a frequency near three-quarters of 2Δ. For the strong-coupling metals involved in their experiments, we may take a value of $g_0 = 0.5$. In Fig. 9 the particle-hole excitation energies are plotted as a function of $(-g_2)$ in the $q = 0$ case for this value of g_0. We see that an excitation given by $x = \frac{3}{4}$ occurs for a value of $g_2 \simeq -0.75$. With these values of g_0 and g_2 the ratio $R_S(\omega)/R_\infty(\omega)$ becomes

$$\frac{R_s(\omega)}{R_\infty(\omega)} = \frac{3\pi}{10}\left(\xi_0 q(\omega)\right)^2 \frac{1}{(2\sqrt{3})^{1/3}}\left(\frac{\lambda_L}{\xi_0}\right)^{\frac{4}{3}} \frac{1}{x^{11/3}} \frac{1}{1-x^2} \times$$

$$\left\{-\frac{5}{2}x^2(1-x^2) + x\left(5-\frac{3}{4}x^2\right)(1-x^2)^{\frac{1}{2}} \operatorname{Arcsin} x\right.$$

$$\left. - \left(\frac{5}{2}+x^2\right)(\operatorname{Arcsin} x)^2\right\} \left|\frac{\partial P}{\partial(\xi_0 q)}\right|^{-1}_{q=q(\omega)}. \qquad (5.25)$$

This ratio has been evaluated as a function of $x = \hbar\omega/2\Delta$, and the results are indicated in Fig. 10. We see that our results do indicate the presence of absorption due to the creation of particle-hole excitons. We do not observe the peak-like structure of Tsuneto's results since our approximations are not valid beyond the regions plotted. However, as in the case of Tsuneto's work, our absorption is at least an order of magnitude too small to explain the observed experimental data. For example, with ξ_0/λ_L equal to 4 for lead, our results indicate an absorption ratio of about 0.003 at x = 0.825, while the observed value is of order 0.1. Therefore, we can conclude that the particle-hole exciton absorption cannot explain the lead and tin precursor absorption observed by Ginsberg et al.,[7] within the approximations of this weak coupling calculation.

It should be noted that our work, as well as that of Tsuneto, has been based on a bulk sample calculation while the experiments have been performed on thin films. The wave vectors, q, of importance are of the same order as the thickness of the films. Therefore, an

improved calculation might be one in which the original unperturbed wave functions are not three dimensional plane waves, but ones in which the finite thickness of the film is taken into account, i.e., a wave function which is an infinite plane wave in two dimensions and sinusoidal in the third with boundary conditions appropriate to the thickness of the film taken into account.

VI. CONCLUSIONS

While we have approximated the L^{th} spherical harmonic of the two-body interaction by a separable potential, $V_L(\vec{k},\vec{k}') = -V_L$ for $|\epsilon_k|$, $|\epsilon_{k'}| < \hbar\omega_c$ and zero, otherwise, in general, if the potential is independent of crystallographic orientation, the numbers L and M remain good quantum numbers for the excitations in the limit $\vec{q} \to 0$. For a nonseparable potential, i.e., if $V_L(\vec{k},\vec{k}')$ is not of the form $\varphi_L(\vec{k})\,\varphi_L^*(\vec{k}')$, there may be more than one exciton state for a given L and M. While the excitons should give a negligible contribution to the specific heat, it may prove possible to observe the thermally-excited odd L excitons (spin waves) by magnetic resonance techniques. The precursor infrared absorption observed in Pb and Hg cannot be explained within the framework of a bulk sample calculation. It would be interesting to carry out an explicit calculation of the absorption coefficient for a thin film geometry in an attempt to reconcile the difference between the theoretical predictions and experiment.

Fig. 1 The two vertices occurring in the full equations of motion for $b_k^\dagger(\vec{q})$. In the linearized equations only vertices with certain values of $\vec{p}'$ and $\vec{q}'$ are retained.

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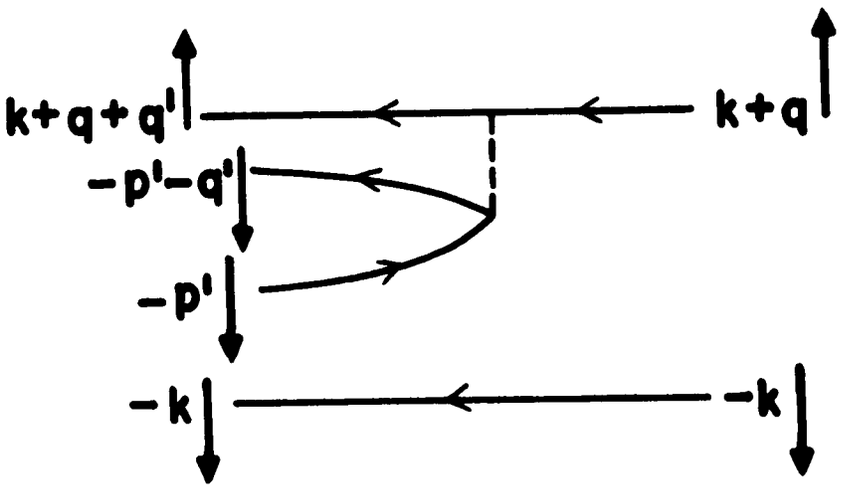
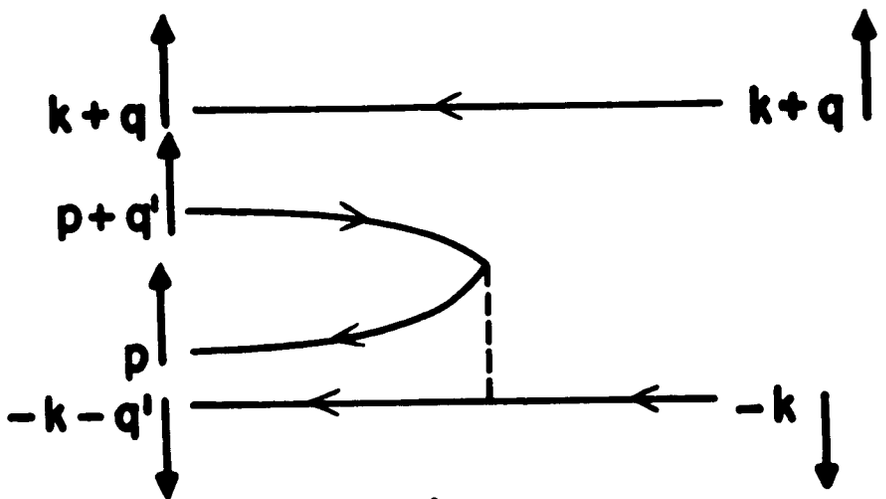


a



b

Fig. 2 The vertices retained in the full linearized equation of motion for $b_k^{\dagger}(\vec{q})$. Vertices f, g, h, and i were neglected by Anderson and by Rickayzen. The particle-hole excitons are obtained only if the interactions shown in h and i are included.

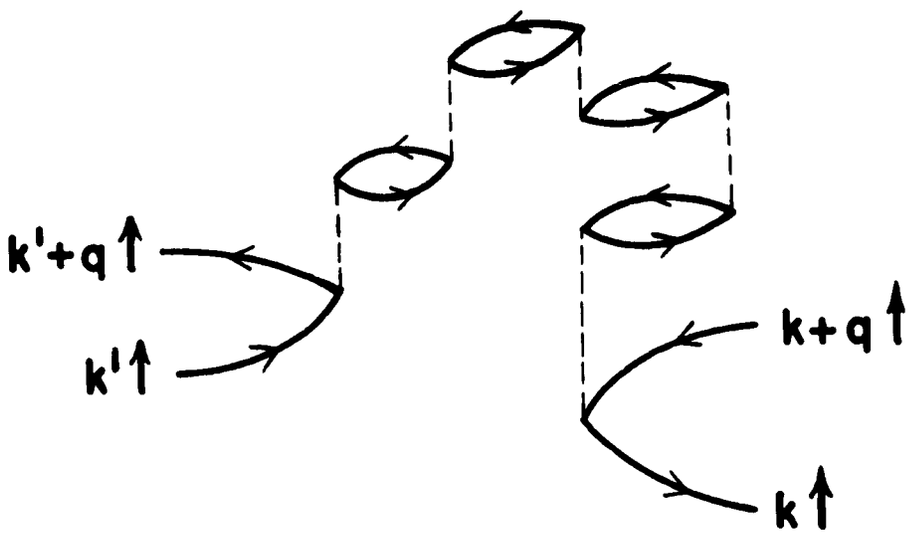
a
b
c
d
e
f
g
h
i

Fig. 3 The vertices retained in the full linearized equation of motion for $\rho_{k\uparrow}(\vec{q})$. Vertices a, e, and f were neglected by Anderson and Rickayzen.

a
b
c
d
e
f
g
h

Fig. 4 A typical diagram retained within the random-phase approximation to $\rho_{k\sigma}(\vec{q})$ in the normal state.

Fig. 5 The random-phase approximation to the screened interaction line.


k'+q
k'
k+q
k

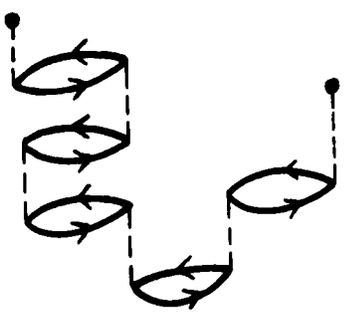


Fig. 6 The L-state exciton energy in the limit $q \to 0$ as a function of the L-wave coupling constant g_L, where s-state pairing in the ground state has been assumed. The solid curve is based on the Anderson-Rickayzen equations while the slightly higher dashed curve includes the effect of the vertices shown in Figs. 2(h) and 2(i) for $g_0 = 0.25$. For $g_L > g_0$ the L-state exciton energy is imaginary. If g_L is the largest coupling constant, the linearization should be carried out with respect to L-state pairing in the ground state.

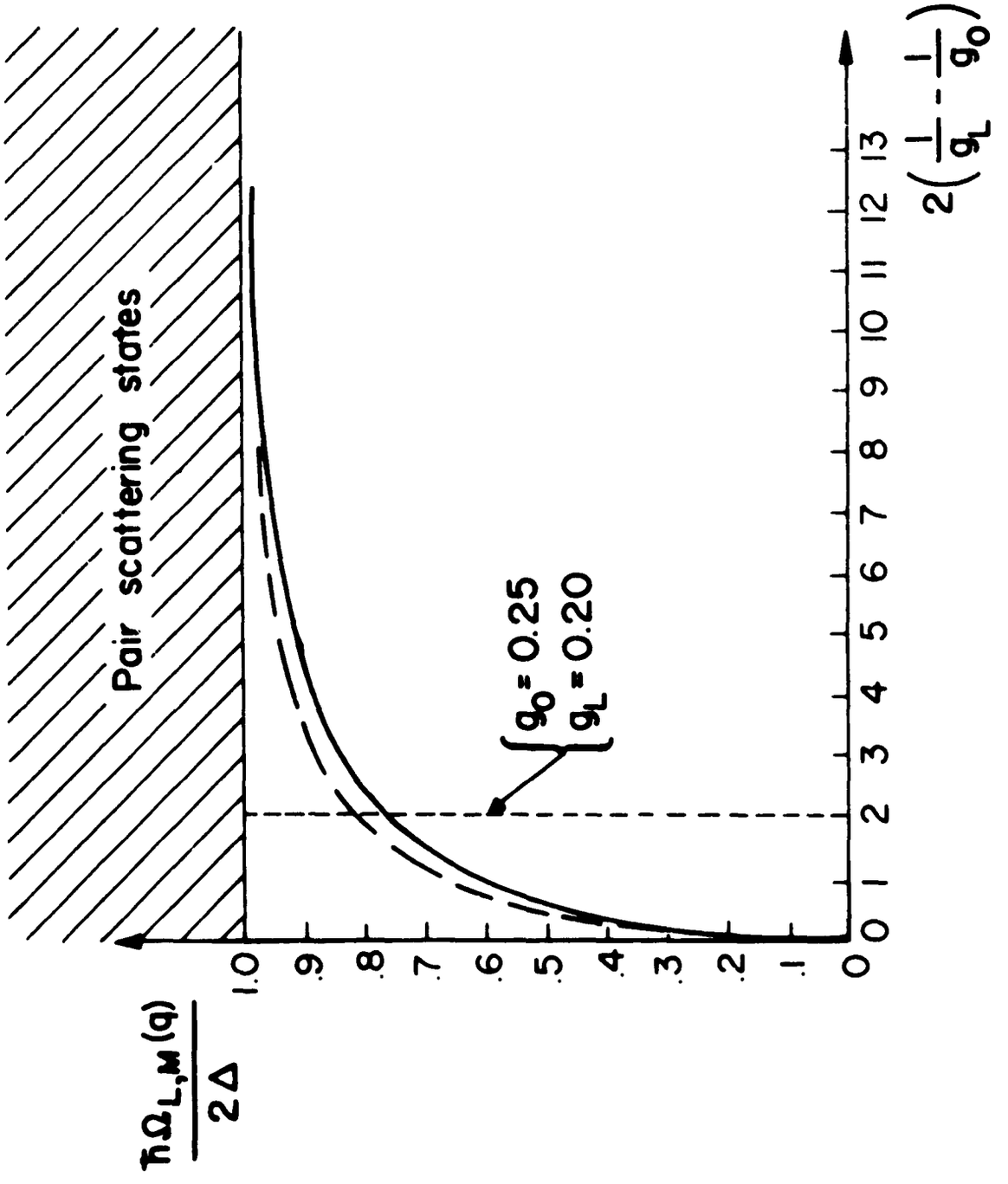

Pair scattering states
$g_0 = 0.25$
$g_L = 0.20$
$\frac{\hbar\Omega_{L,M}(q)}{2\Delta}$
$2\left(\frac{1}{g_L} - \frac{1}{g_0}\right)$
0
1
2
3
4
5
6
7
8
9
10
11
12
13
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0

Fig. 7 The $\vec{p}$-state exciton energy as a function of momentum $\vec{q}$ for $g_0 = 0.25$ and $g_1 = 0.24$ or 0.25. The parameter ξ_0 is the coherence length $\sim 10^{-4}$ cm. Notice that the exciton states are strongly bound only for $q^{-1} > \xi_0$.

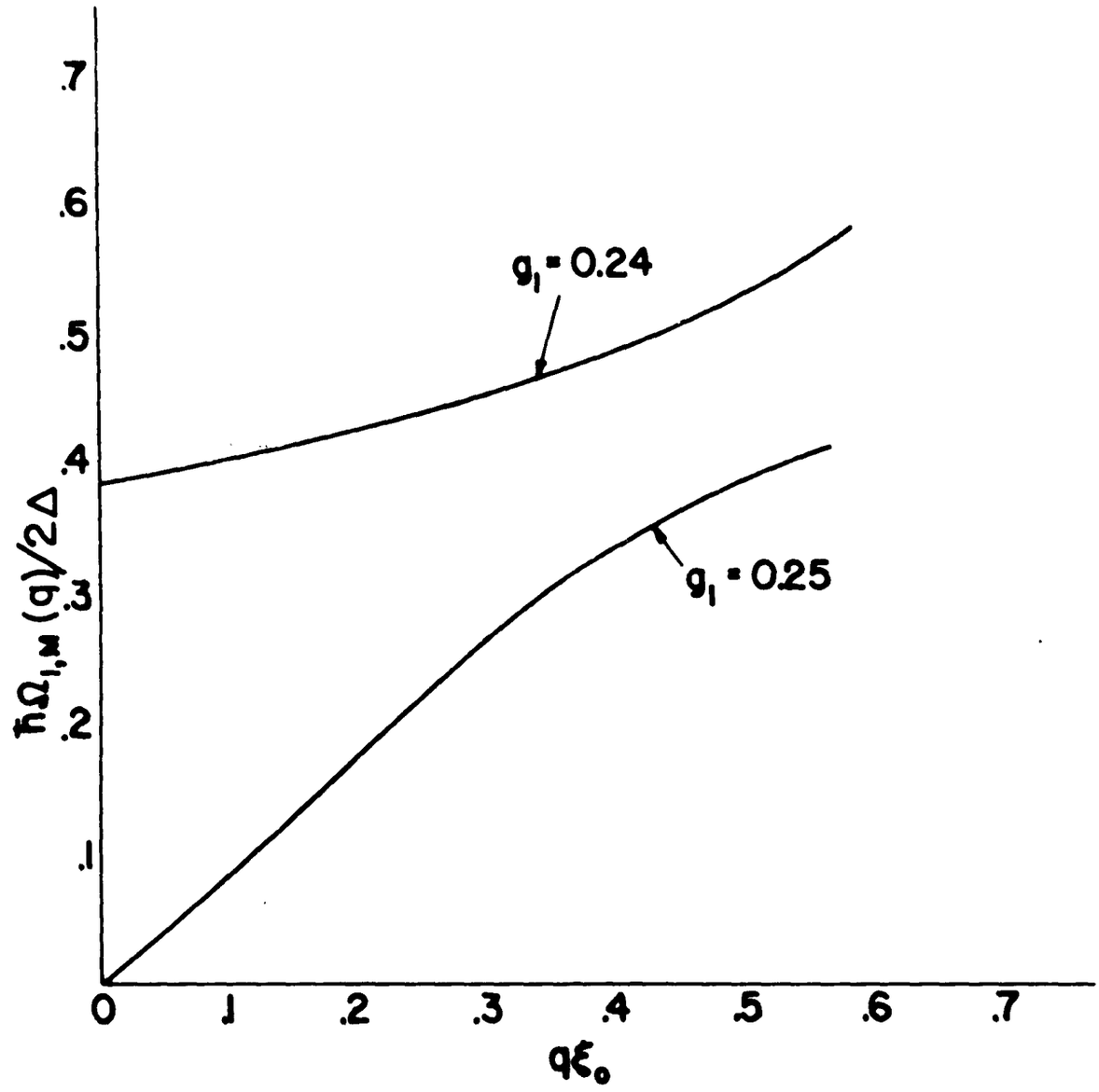

.7
.6
.5
.4
.3
.2
.1
$\hbar\Omega_{1,M}(q)/2\Delta$
$g_1 = 0.24$
$g_1 = 0.25$
0
.1
.2
.3
.4
.5
.6
.7
$q\xi_0$

Fig. 8 The energy of the L-state particle-hole exciton as a function of the L-wave coupling constant g_L with $g_0 = 0.25$. For $g_L > 0$ the particle-particle exciton described by Figs. 6 and 7 is bound while for $g_L < 0$ the particle-hole exciton is bound. In the absence of the direct interaction V_D, the s-state exciton is essentially a bound particle-particle (and hole-hole) pair. With the inclusion of long-range Coulomb interactions, the s-state exciton becomes a plasmon described as a particle-hole pair.

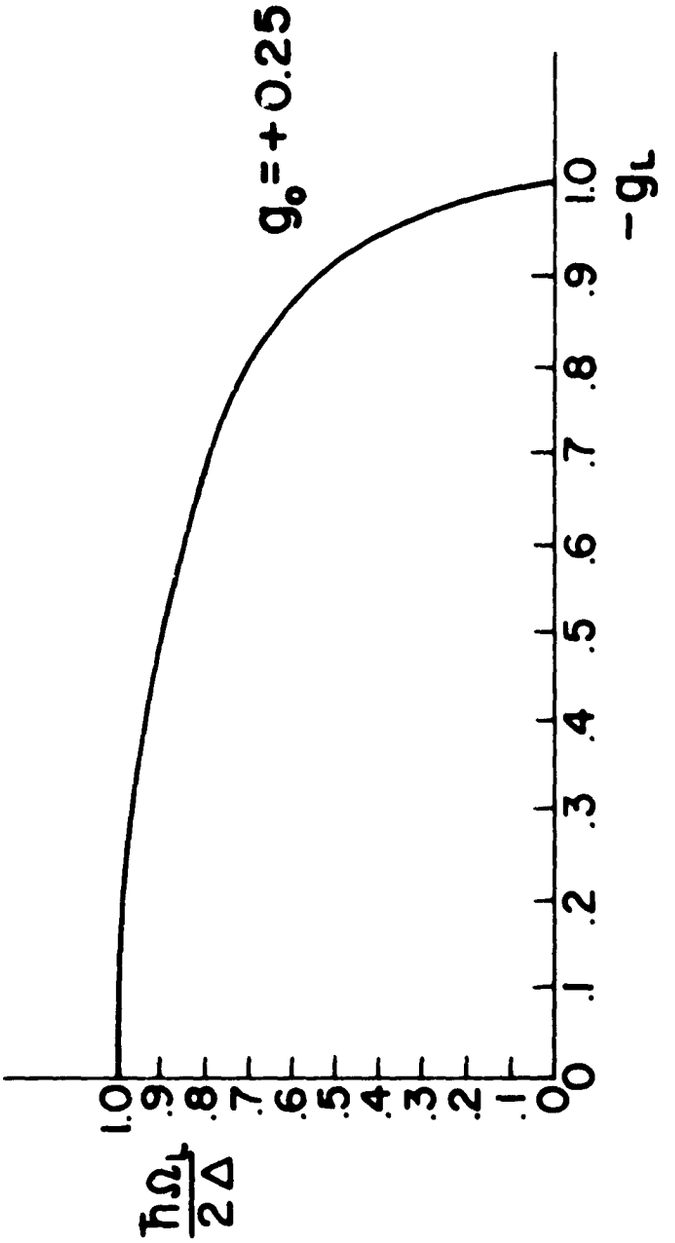

$g_o = +0.25$
$-g_L$
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
$\frac{\hbar\Omega_\perp}{2\Delta}$
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0

Fig. 9 The energy of the D-state particle-hole exciton, in the limit of $q \to 0$, as a function of the D-wave coupling constant g_2 with $g_0 = 0.50$.

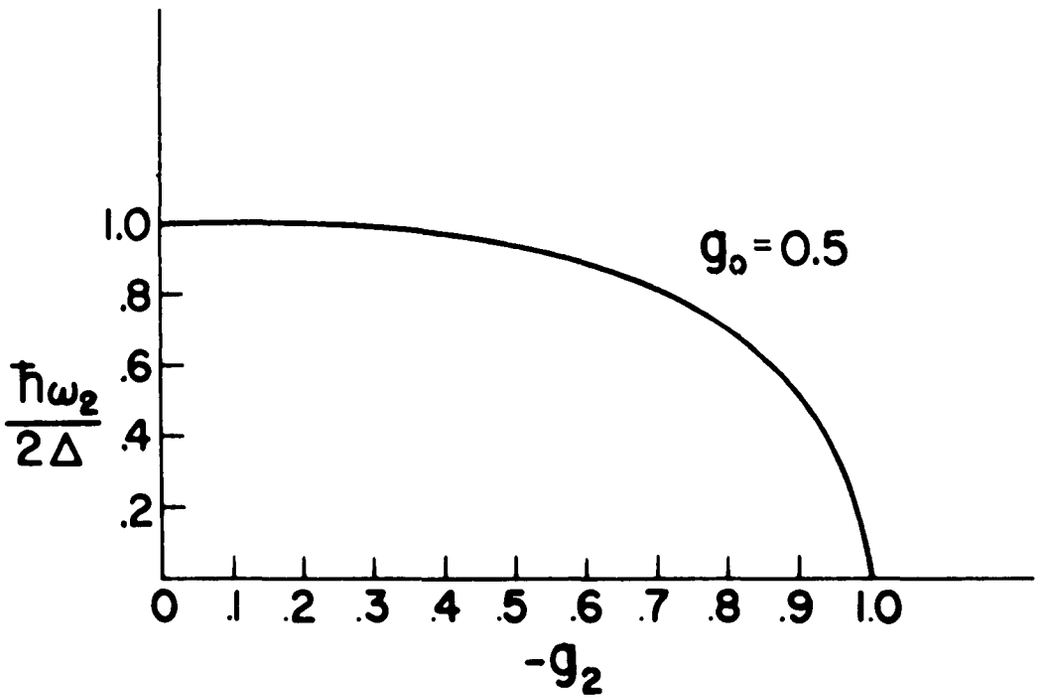
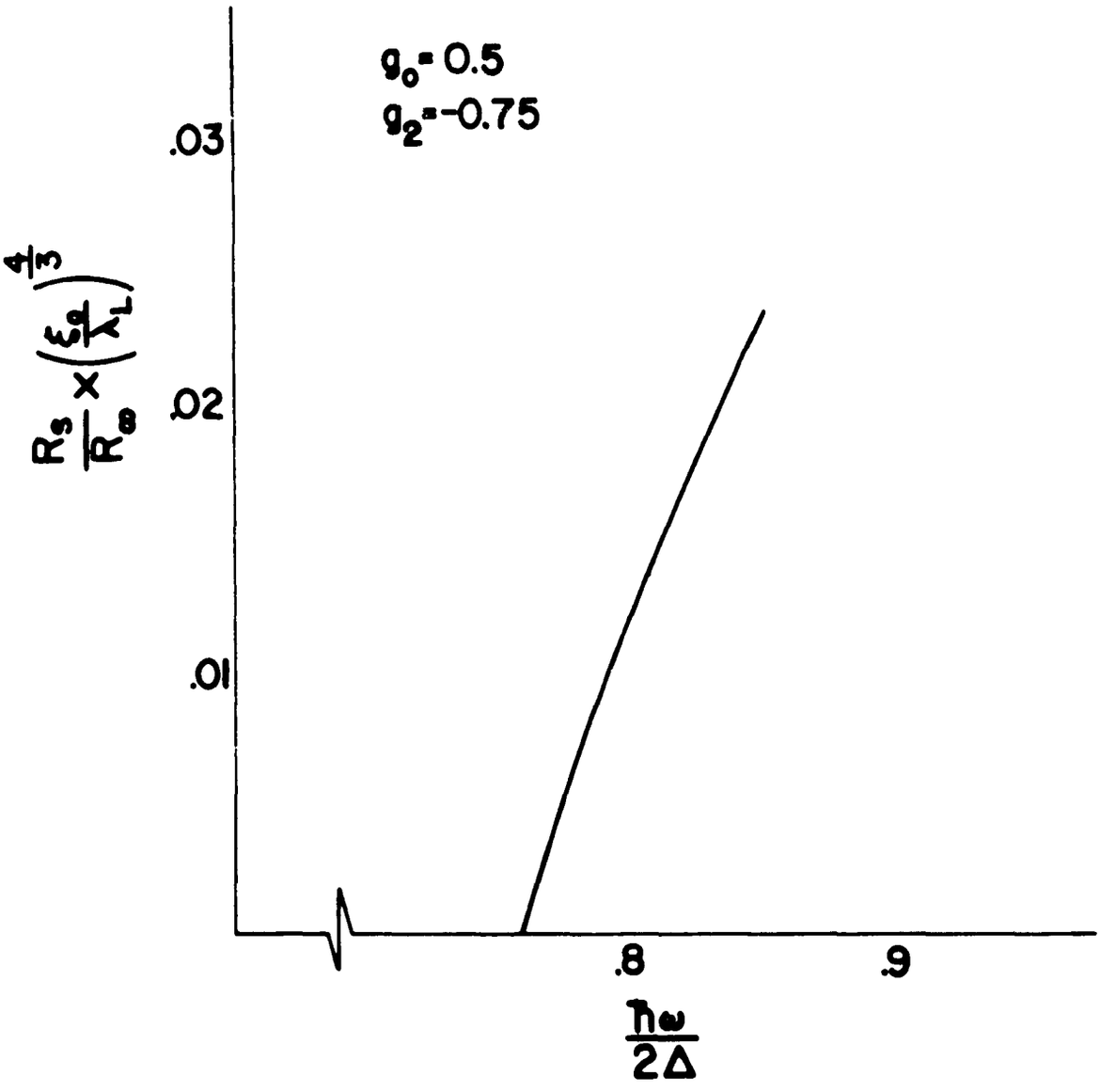

1.0
.8
.6
.4
.2
$\frac{\hbar\omega_2}{2\Delta}$
$g_0 = 0.5$
0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
$-g_2$

Fig. 10 The ratio of the surface resistance of a superconductor due to creation of D-state particle-hole excitons to that of a normal metal in the extreme anomalous limit calculated for the D-wave coupling constant $g_2 = -0.75$ and $g_0 = 0.50$.


$g_0 = 0.5$
$g_2 = -0.75$
.03
.02
.01
$\frac{R_s}{R_n} \times \left(\frac{\xi_0 q}{\lambda_L}\right)^{4/3}$
.8
.9
$\frac{\hbar\omega}{2\Delta}$

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Part II

Lifetime Effects in Condensed Fermion Systems

I. THE ROLE OF LIFETIME EFFECTS

In a many body system a particle in an excited state does not remain there indefinitely due to its interacting with the other elementary excitations of the system. The presence of this lifetime can be taken into account in the description of the excited state by specifying its energy as $E = E_0 + iE_1$, where E_0 and E_1 are both real. From this we see that a well defined excitation energy only has significance when the state is long lived; i.e., when $E_1 \ll E_0$.

The effects of finite lifetimes on the excitation spectrum cannot be investigated easily using an equation of motion approach or a variational one such as that used in the original[1] BCS theory of superconductivity in which exact eigenstates are obtained for a reduced Hamiltonian containing no damping effects. The mathematics of a Green's function formulation provides the most convenient means for investigating the lifetimes. The basis for an analysis of this type is the calculation of the fermion self-energy diagram of Fig. 1 whose imaginary part is related to the damping rate. The solid line represents the fermion and the dotted one is the full interaction line.

Migdal[2] has evaluated the electron self-energy for a normal metal with the interaction line representing the phonon propagator. For excitation energies ϵ_k (measured relative to the Fermi surface) much less than the maximum phonon energy $\hbar\omega_0$ he finds the imaginary part of the self-energy to be

$$E_1(k) = (\hbar\omega_0)\,\frac{\pi\lambda_0(2-\lambda_0)}{6(1-\lambda_0)^2}\,\left(\frac{\epsilon_k}{\hbar\omega_0}\right)^3. \tag{1.1}$$

In this expression, λ_0 is a dimensionless parameter of the order of 1/2 related to the coupling strength between the electrons and phonons. For energies $\epsilon_k \geq \hbar\omega_0$, Migdal finds that the damping rate due to real phonon emission saturates out to a constant value. For the range of excitation energies larger than about ten times the maximum phonon energy, damping due to hole-electron pair production becomes important and has been obtained by Quinn and Ferrell[3] by allowing the interaction line in the self-energy diagram to represent the screened Coulomb interaction. For values of $\epsilon_k < E_F$, the Fermi energy, they obtain

$$E_1(k) = (\hbar\Omega_p)\left(\frac{\pi^2\sqrt{3}}{256}\right)\left(\frac{\epsilon_k}{E_F}\right)^2, \tag{1.2}$$

where $\Omega_p = (4\pi ne^2/m)^{1/2}$ is the plasmon frequency.

In the case of a transition to a superfluid state, the effect of damping on the required energy gap and the transition temperature is closely related to the relative roles the phonon and Coulomb interactions play in constructing a criterion for the possibility of a transition.

In the original BCS theory the general criterion for superconductivity is that the attractive electron-phonon interaction as calculated by Bardeen and Pines[4] dominate the Coulomb interaction for those matrix elements which are important in the superconducting wave

function. The specific model treated by BCS is one in which the two-body interaction is taken to be the separable potential

$$-V_{kk'} = (V_c + V_{ph})_{kk'} = -V < 0 \qquad (1.3)$$

$$\text{for } |\varepsilon_k|, |\varepsilon_{k'}| < \hbar\omega_c \text{ , and}$$

$$-V_{kk'} = 0 \text{ , otherwise.}$$

In this expression, V_c and V_{ph} are the Coulomb and phonon interactions and

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m} - E_F .$$

The BCS criterion has been further investigated by Pines[5] who considers the sign of the interaction $(V_c + V_{ph})_{kk'} = -V_{kk'}$ in the zero excitation energy limit: i.e., when both $|\vec{k}|$ and $|\vec{k}'|$ are taken equal to the Fermi momentum k_F. In this model the ionic charge is considered to be a continuous fluid whose vibrational spectrum provides the frequency distribution of the phonons used in the problem. With this model, Pines finds $V_{kk'}$ will be positive at the Fermi surface if Umklapp processes are considered (which are calculated without including periodicity effects). Using this criterion, that the net interaction must be positive at the Fermi surface, predictions are made as to which elements in the periodic table ought to be superconducting. In the case of those elements for which the neglect of periodicity in calculating the matrix elements may be valid, certain qualitative results are obtained which agree with experiment.

Another investigation into the criterion for superconductivity has been made by Bogoliubov,[6] and Abrikosov and Khalatnikov.[7] In their treatments, the Coulomb and phonon interactions are considered explicitly in order to determine the relative roles played by these two interactions in the criterion. In their method, the attractive potential due to the phonon interaction only is taken into account by inclusion of a separable potential (1.3), and is characterized by the positive coupling constant $g = N(0)V > 0$. The quantity $N(0)$ is the density of Bloch electron states of one spin at the Fermi surface. The Coulomb interaction is accounted for by a constant repulsive potential acting within the momentum band, $2\hbar\omega_c'$, which is larger than the range, $2\hbar\omega_c$, over which the attractive potential acts. This corresponds to the fact that the Coulomb interaction is screened at distances of the order of the lattice spacing and will be explained in greater detail below. The repulsive potential's coupling constant is denoted by $g' = -N(0)V' > 0$. With these definitions the gap, Δ, is the solution of

$$1 = \left(g - \frac{g'}{1 + g' \ln\left(\frac{\omega_c'}{\omega_c}\right)} \right) \ln \frac{2\omega_c}{\Delta} . \qquad (1.4)$$

From this equation we see that the criterion for superconductivity in this model is

$$g > \frac{g'}{1 + g' \ln\left(\frac{\omega_c'}{\omega_c}\right)} , \qquad (1.5)$$

replacing the condition $g > g'$ given by the separable potential model

of BCS. The role of the Coulomb interaction is thus reduced by the factor $\ln(\omega_c'/\omega_c)$ which is typically about 5 if $\hbar\omega_c' \simeq E_F$. Furthermore, when

$$g' > g > \frac{g'}{1 + g' \ln\left(\frac{\omega_c'}{\omega_c}\right)} \tag{1.6}$$

the net potential may be everywhere repulsive in momentum space and still lead to a superconducting state. This result may be more clearly understood if we consider the relationship between an interaction in coordinate space and its momentum space Fourier transform. For the limiting case of a constant potential having infinite extent in momentum space, the coordinate space interaction takes the form of an infinitely sharp delta function. As the extent of the interaction in momentum space is shortened, the corresponding delta function broadens increasing the interaction's extent in coordinate space. We can now apply this notion to the Bogoliubov potential in which the Coulomb interaction has a greater extent in momentum space than the phonon interaction. With $\omega_c' > \omega_c$, the coordinate space extent of the screened Coulomb interaction (the order of a lattice parameter) is less than that of the phonon interaction. From the BCS theory we know that the superconducting state is characterized by the formation of bound pairs of electrons whose relative coordinate wave function has an extent of $\xi_0 \simeq 10^{-4}$ cm. Since a lattice parameter is roughly 10^{-8} cm, the bound pair of electrons rarely experiences the Coulomb potential, although its effects may not be totally neglected since it is of large magnitude. The bound state is mainly determined by the attractive phonon interaction even though

its average magnitude may be less than that due to Coulomb effects. In this way a net repulsive interaction in momentum space can lead to a superconducting transition. Of course, when the range of the two interactions is the same so that $\omega_c' = \omega_c$, the Bogoliubov criterion reduces to that of Pines and the model solved by BCS. One difficulty in greatly reducing the Coulomb effects is that the critical temperature increases so that most metals might be expected to exhibit superconductivity.

Bardeen[8] has suggested that lifetime effects due to the Coulomb excitation of particles out of the Fermi sea could be used to determine the cutoff, $\hbar\omega_c'$, for the Coulomb interaction. When the energy ϵ_k is sufficiently large, the excitation may decay so rapidly that it is not well defined. For a rough estimate of Coulomb damping effects we can use the Quinn and Ferrell result of equation (1.2). From this equation we see that for $\epsilon_k \simeq E_F$, the imaginary part of the self energy is

$$E_1(k \simeq k_F) \simeq \frac{2}{3} E_F . \tag{1.7}$$

As we will see in the next section, damping begins to affect the gap equation when $E_1(k)$ and ϵ_k are equal. Although equation (2.1) does not hold for $\epsilon_k > E_F$, the result (1.7) indicates that the desired equality would not take place until $\epsilon_k > E_F$. But for these values of ϵ_k the matrix elements of the screened Coulomb interaction are small, indicating that Coulomb damping effects play a small role in determining the cutoff $\hbar\omega_c'$.

This result shows that the inclusion of Coulomb damping effects cannot reduce the logarithm term in the Bogoliubov result (1.5). This reduction is necessary if the role the Coulomb interaction plays in establishing a criterion is to be increased. On the other hand, as a consequence of (1.5), most metals would be expected to be superconducting. We can only conclude that a careful examination of periodicity effects in the electron-phonon interaction would be necessary in order to establish a satisfactory criterion for superconductivity. This problem is not considered in this thesis.

Below, we outline a formalism for working with damping effects in a fermion system and consider the consequences of certain assumed forms for this damping. We also consider the problem of He^3 for which no superconducting transition has been observed in experiments down to $T \geq 5 \times 10^{-3}$°K, although theoretical predictions neglecting damping have predicted a critical temperature an order of magnitude larger than this value. We find that the damping of excited states will tend to reduce the net attractive interaction necessitating a corresponding lowering of the critical temperature.

II. EVALUATION OF THE SELF-ENERGY DIAGRAM

In calculating the self energy diagram of Fig. 1 we will follow most closely the formulations of Nambu[9] and Eliashberg[10]. It is useful to introduce the two-component form for the electron wave fields

$$\Psi_p = \begin{pmatrix} c_{p\uparrow} \\ c^{\dagger}_{-p\downarrow} \end{pmatrix}, \tag{2.1}$$

along with the Pauli spin matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

The operators $C^{+}_{p\sigma}$ and $C_{p\sigma}$ are the usual electron creation and annihilation operators whose only non-vanishing anti-commutators are

$$\{C_{p\sigma}, C^{+}_{p'\sigma'}\} = \delta_{pp'}\delta_{\sigma\sigma'}, \tag{2.3a}$$

from which we see that

$$\{(\Psi_p)_i, (\Psi^{+}_{p'})_j\} = \delta_{pp'}\delta_{ij}. \tag{2.3b}$$

In evaluation of Feynman diagrams within the new notation of (2.1) and (2.2), all the usual rules hold with the Ψ's taking the roles of the c's of the old notation. The only new rule is the addition of a factor of τ_3 at each vertex. This can be seen, for example, by considering the familar kinetic energy and electron-phonon interaction Hamiltonians

$$H_K = \sum_{p,\sigma} \varepsilon_p C^{+}_{p\sigma} C_{p\sigma}, \tag{2.4a}$$

and

$$H_{el\text{-}ph} = \sum_{p,q,\sigma} g_q C^{+}_{p+q\sigma} C_{p\sigma} (a_q + a^{+}_{-q}), \tag{2.4b}$$

where $\epsilon_p = \frac{\hbar^2 p^2}{2m} - E_F$, g_q is the strength of the interaction, and a_q is a bare phonon operator. In our new notation equations (2.4) become

$$H_K = \sum_{p} \varepsilon_p \left[\Psi^{+}_{p} \tau_3 \Psi_p + 1 \right], \tag{2.5a}$$

and

$$H_{el\text{-}ph} = \sum_{p,q} g_q \Psi^+_{p+q} \tau_3 \Psi_p (a_q + a^+_{-q}). \tag{2.5b}$$

If we consider the Fermi sea as our vacuum, $|0\rangle$, we define the Green's function for free electrons ($H = H_K$) by

$$\langle 0|T\{\Psi_i(\vec{x},t), \Psi^+_j(\vec{x}',t')\}|0\rangle = G^0(\vec{x}-\vec{x}', t-t')|_{ij}, \tag{2.6a}$$

where T is the time ordering operator. Taking the Fourier transform of (2.6a), it follows that

$$G^0_p = \frac{i}{p_0 - (\varepsilon_p + i\eta_p)\tau_3}, \tag{2.6b}$$

where $p = (\vec{p}, p_0)$. The Green's function, G_p, for the total Hamiltonian $H = H_K + H'$ is related to the self energy, Σ_p, and the lowest order Green's function (2.6b) through the Dyson equation

$$G_p = (G^0_p)^{-1} - \Sigma_p. \tag{2.7}$$

The most general form for Σ_p would be one having components along all four orthogonal axes; i.e.,

$$\Sigma_p = -i(\zeta_p + \chi_p \tau_3 + \Delta_p \tau_1 + \bar{\Delta}_p \tau_2). \tag{2.8}$$

However, the first order Hamiltonian (2.5a) is invariant under rotations in a plane orthogonal to the unit matrix and τ_3. Therefore, we can set $\bar{\Delta}_p = 0$ in (2.8) and take

$$\Sigma_p = -i(\zeta_p + \chi_p \tau_3 + \Delta_p \tau_1). \tag{2.9}$$

This equation, together with (2.6b) and (2.7), gives

$$G_p = \left(\frac{i}{Z_p}\right) \frac{p_0 + \frac{1}{Z_p}(\xi_p + \chi_p)\zeta_3 + \frac{1}{Z_p}\Delta_p \zeta_1}{p_0^2 - \frac{1}{Z_p^2}\left[(\xi_p^2 + \chi_p^2) + \Delta_p^2\right] + i\eta_p}, \tag{2.10}$$

where

$$Z_p = 1 - (\zeta_{3p}/p_0). \tag{2.11}$$

The Dyson equation connecting the self-energy, Σ_p, with the exact Green's function for the problem and the total vertex, $\Gamma_{pp'}$, is

$$\Sigma_p = -i \int \frac{d^4p'}{(2\pi)^4} \Gamma^0_{pp'} G_{p'} D_{pp'} \Gamma_{pp'}, \tag{2.12a}$$

where $D_{pp'}$ is the propagator for the interaction. The self-energy is generally obtained from a perturbation expansion. In the case of an electron-phonon interaction, $D_{pp'}$ is the phonon propagator and $\Gamma^0_{p'p}$ represents the coupling constant. For this case Migdal[2] has shown that $\Gamma_{p'p} \simeq \Gamma^0_{p'p}$ to order (m/M), where M is the ion mass. If the vertex function for any interaction is approximated by its zeroth order value, this is equivalent to summing a perturbation series for the self-energy only involving diagrams in which no two interaction lines cross. The result of this amounts to evaluating the self-energy diagram of Fig. 1 in which the solid line represents the full Green's function for the problem and the dotted one the full interaction. For a general potential, $V_{pp'}$, this approximation to equation (2.12a) is expressed by

$$\Sigma_p = -i \int \frac{d^4p'}{(2\pi)^4} \zeta_3 G(p') \zeta_3 V_{p'p} \tag{2.12b}$$

where the τ's appear for the reasons discussed previously. Substituting (2.10) into (2.12b) and considering the τ_1 component of the result, we obtain the gap equation

$$\Delta_p = -i\int \frac{d^4p'}{(2\pi)^4} \frac{1}{(Z_{p'})^2} \frac{\Delta_{p'} V_{p'p}}{(p_0')^2 - \frac{1}{Z_{p'}^2}\left[(\xi_{p'}+\chi_{p'})^2 + \Delta_{p'}^2\right] + i\eta_{p'}} . \quad (2.13)$$

The integration over p_0' can be carried out by the considerations of Eliashberg.[10] The Green's function $G_{p'}$, when considered as a function of p_0, is analytic in the upper (lower) half plane for $p_0 > (<) 0$. Therefore, the $i\eta_{p'}$ term may be dropped in the integrand when the p_0' integration is taken along the contour C_1 of Fig. 2. We will now assume that the potential $V_{pp'}$ is real and independent of p_0 and p_0'. With this condition, consideration of the complex function $\bar{G}(p,z)$ which coincides with $G(p,p_0)$ on the upper (lower) half plane for $p_0 > (<) 0$ allows the contour C_1 to be deformed to C_2 of Fig. 3. Since the values of $i\bar{G}(p,z)$ on the opposite sides of the cut $(0,\infty)$ are complex conjugates, we obtain

$$\Delta_p = 2\,\mathrm{Im}\int \frac{d^3p'}{(2\pi)^3} V_{p'p} \int_0^\infty \frac{dp_0'}{(2\pi)} \frac{\Delta_{p'}/Z_{p'}^2}{(p_0')^2 - \bar{E}_{p'}^2} , \quad (2.14)$$

where

$$\bar{E}_{p'}^2 = \frac{1}{(Z_{p'})^2}\left[\xi_{p'}^2 + \Delta_{p'}^2\right] = \left(\frac{E_{p'}}{Z_{p'}}\right)^2 \quad (2.15)$$

Since the integrand is symmetric in p_0' the lower limit in (2.14) may

be replaced by $-\infty$ and the result multiplied by 1/2. Assuming that the analytic continuation of G_p into the lower half plane for $p_0 > 0$ has a simple pole corresponding to an elementary excitation of energy

$$\bar{E}_{p'}^{r} = \bar{E}(\vec{p}', p_0')\Big|_{p_0' = \bar{E}_{p'}^{r}}, \tag{2.16}$$

the integral equation becomes

$$\Delta_p = -Re \int \frac{d^3p'}{(2\pi)^3} \frac{V_{p'p}\ \Delta(\vec{p}', \bar{E}_{p'}^{r})}{2 Z_{p'} E_{p'} \left\{1 - E_{p'} \left[\partial (1/Z_{p'})/\partial p_0'\right]\right\}_{p_0' = \bar{E}_{p'}^{r}}}. \tag{2.17}$$

In this expression the quantity ζ_p, related to Z_p through (2.12), satisfies an integral equation similar to (2.13) obtained by considering the unit matrix element component of equation (2.8) and gives the normal state self-energy in the limit of $\Delta_p \to 0$. The imaginary part of ζ_p gives rise to damping effects in the energy gap equation. When ζ_p is set equal to zero ($Z_p = 1$), the integrand of (2.17) becomes real and the equation reduces to that of BCS.

III. DAMPING EFFECTS FOR ENERGY INDEPENDENT $V_{p'p}$

We will investigate the effects of damping on equation (2.17) by considering various forms of the function ζ_p. The real part of ζ_p will not be needed since it serves to renormalize the excitation energies ϵ_p and the interaction $V_{pp'}$. It is the imaginary part of ζ_p which provides for the damping effects. We consider a general form for Z_p in which the damping is proportional to a power of p_0; i.e.,

$$Z_p = 1 + \frac{i\alpha |p_0|^n \operatorname{sgn}(p_0)}{p_0}. \tag{3.1}$$

It should be noted here that our assumed form for ζ_p's imaginary part is not entirely satisfactory since this quantity must vanish for $|p_0| < \Delta_p$. Therefore, for n = 0 in (3.1) we may only consider the $\Delta_p \to 0$ situation. For n = 1 the approximation is much better, and for the case where n = 2, the error introduced is entirely negligible. For $n \neq 0$, substitution of (3.1) into (2.17) gives

$$\Delta_p = -\int \frac{d^3p}{(2\pi)^3} V_{pp} \frac{\Delta_{p'}}{2E_{p'}\left[1+\hbar^2\alpha^2|p_0'|^{2(n-1)}\right]}, \qquad (3.2)$$

where p_0' is the solution to $p_0' = \bar{E}_{p'}^{\,r}$.

We will consider the three cases n = 0, 1, and 2.

(1) n = 0

The n = 0 case corresponds to constant damping. For the separable two body interaction

$$V_{pp'} = -V, \ |\varepsilon_{p'}|, |\varepsilon_p| < \hbar\omega_c,$$
$$V_{pp'} = 0, \text{ otherwise}, \qquad (3.3)$$

setting $\Delta_p = 0$ in the n = 0 integral equation allows us to determine the minimum value of α such that no transition to the superconducting state occurs; i.e., $T_c = 0^\circ K$. Although equation (3.2) does not hold for n = 0, it is easy to show that the $\Delta_p = 0$ equation for this case is

$$\frac{1}{N(0)V} = \int_0^{\hbar\omega_c} \frac{\varepsilon\, d\varepsilon}{\varepsilon^2+\alpha_c^2}, \qquad (3.4)$$

where N(0) is the density of Bloch states of one spin per unit energy

at the Fermi surface. The solution to (3.4) gives the critical value for α,

$$\alpha_c = \hbar\omega_c \exp[-1/N(0)V]. \tag{3.5}$$

This result agrees with that of Suhl.[11]

(2) n = 1

With the separable potential of case (1), the gap equation (3.2) gives

$$\frac{1}{N(0)V} = \int_0^{\hbar\omega_c} d\xi \, \frac{1}{(\xi^2+\Delta^2)^{1/2}(1+\alpha^2)} . \tag{3.6}$$

The solution is of the BCS type with

$$\Delta_p = \hbar\omega_c / \sinh[(1+\alpha^2)N(0)V], \text{ for } |\xi_p| \le \hbar\omega_c ; \tag{3.7}$$

$\Delta_p = 0$, otherwise.

Thus, the inclusion of linear damping modifies the BCS solution only through the introduction of a reduced effective coupling constant $N(0)V/(1+\alpha^2)$.

(3) n = 2

For the case of quadratic damping, the $T = 0^{\circ}K$ gap equation is

$$\Delta_p = -\frac{1}{2}\int \frac{d^3p'}{(2\pi)^3} \frac{\Delta_{p'} V_{pp'}/2}{[(\xi_{p'})^2+(\Delta_{p'})^2]^{1/2}\left\{\left[1+4\alpha^2((\xi_{p'})^2+(\Delta_{p'})^2)\right]^{1/2}-\frac{1}{2}\right\}} . \tag{3.8}$$

From the result of (1.7) we have seen that the screened Coulomb

interaction leads to $1/\alpha > E_F$ so that the effects of Coulomb damping on equation (3.8) are negligible since the screened Coulomb matrix elements, $V_{pp'}$, are small for energies $\epsilon_p > E_F$. Therefore, inclusion of Coulomb damping cannot serve to decrease the extent of the Coulomb interaction in (3.8). Consequently, the parameter ω_c' in the Bogoliubov criterion (1.5) for a superconducting state in metals is essentially unchanged from its value computed in the absence of damping.

For He^3, damping effects are important in determining Δ_p and the transition temperature, T_c, for a possible superfluid state. Due to the hard-core potential, the integral equation must be solved in coordinate space. However, we can obtain an estimate for the reduction of T_c due to damping effects by considering an effective separable potential given by (3.3). With this potential, the transition temperature is given by

$$\frac{1}{N(0)V} = \int_0^{\hbar\omega_c} d\xi \, \frac{\tanh(\beta_c \xi/2)}{\xi\left[2(1+4\alpha^2\xi^2)^{1/2}-1\right]}, \tag{3.9}$$

where $\beta_c = (kT_c)^{-1}$.

We will now calculate an approximate value for α. For the thermal conductivity of He^3, Abrikosov and Khalatnikov[12] obtain the theoretical expression

$$K = \frac{8}{3}\frac{\pi^2\hbar^3 p_F^3}{(m^*)^4 T}\left\{\left[\frac{w(\theta,\varphi)(1-\cos\theta)}{\cos(\theta/2)}\right]_{\text{Average}}\right\}^{-1}, \tag{3.10}$$

where p_F is the Fermi momentum and m* is the effective mass of a He^3

atom. The quantity $\omega(\theta,\varphi)$ represents the transition rate for the scattering of a quasi-particle and is related to the scattering amplitude, $f(\theta,\varphi)$, by the usual "golden rule" formula

$$w(\theta,\varphi) = \frac{2\pi}{\hbar} |f(\theta,\varphi)|^2 . \tag{3.11}$$

Since we have chosen an angularly independent separable potential it is consistent to consider $\omega(\theta,\varphi)$ as constant in the expressions (3.10) and (3.11). With this assumption, performing the angular average in (3.10) gives

$$w = \frac{\pi^2 \hbar^3 p_F^3}{(m^*)^4 KT} . \tag{3.12}$$

At this point we consider the above quoted work of Quinn and Ferrell,[3] in which the imaginary part, $E_1(p)$, of the excitation energy is obtained for electrons interacting under the screened Coulomb potential. Their results may be used for the He^3 problem if the screened Coulomb matrix elements are replaced by the scattering amplitude, f. This straightforward generalization gives

$$E_1(p) = \frac{3}{8\pi} \left(\frac{m^*}{\hbar p_F}\right)^3 n f^2 \xi_p^2 , \tag{3.13}$$

where n is the He^3 particle density. Substituting f^2 obtained from (3.11) and (3.12) into (3.13) gives

$$E_1(p) = \frac{3}{16} \left(\frac{n\hbar}{m^* KT}\right) \xi_p^2 . \tag{3.14}$$

Use is now made of the experimental work of Anderson, Salinger and Wheatley[13] who obtain

$$K = 51/T \text{ ergs/cm.sec}^{\circ}K. \tag{3.15}$$

With this value for the thermal conductivity, equation (3.14) becomes

$$E_1(p) = \alpha \varepsilon_p^2 = \frac{3}{16}\left(\frac{nk}{51m^*}\right)\varepsilon_p^2. \tag{3.16}$$

The values $m^* = 2.82m$ and $n = 1.64 \times 10^{22}$ particles/cm^3 give the damping constant

$$\alpha = 4.27 \times 10^{15}(\text{ergs})^{-1} \tag{3.17}$$

When compared to the He^3 Fermi temperature of about $3^{\circ}K$, $\alpha = 1.77\, E_F^{-1}$, indicating that the imaginary and real parts of the excitation energy are equal at $\epsilon_p \simeq E_F/2$. Having arrived at a value of α, we may go on to compute T_c.

The integral of equation (3.9) is approximated by

$$\frac{1}{N(0)V} \simeq \int_0^{\varepsilon_1} d\varepsilon\, \frac{\tanh(\beta_c\varepsilon/2)}{\varepsilon} + \int_{\varepsilon_1}^{\hbar\omega_c} d\varepsilon\, \frac{1}{\varepsilon\left[2(1+4\alpha^2\varepsilon^2)^{1/2}-1\right]} = I_1 + I_2. \tag{3.18}$$

For this approximation to be reasonable we need a value of ϵ_1 for which $4\alpha^2\epsilon^2 \ll 1$ for $\epsilon \leq \epsilon_1$. In this way damping effects are negligible in the ϵ range of I_1 and the hyperbolic tangent may be replaced by unity in I_2. We note that for $\epsilon_1 = 1/10\,\alpha$ the condition of I_1 is satisfied. Furthermore, if we take $T_c^{\circ} = 0.05^{\circ}K$ for the critical temperature without damping as calculated by Emery and Sessler,[14] our choice of ϵ_1 gives $\tanh(\beta_c^{\circ}\,\epsilon_1/2) \simeq 0.91$. Since

our result for T_c will be less than T_c^o, setting $\epsilon_1 = 1/10\ \alpha$ also satisfies the requirement of I_2.

The I_1 integral is identical to that arising in the BCS critical temperature equation. Their numerical evaluation gives

$$I_1 = \log\left(\frac{1.14\mathcal{E}_1}{kT_c}\right). \tag{3.19}$$

For $\epsilon_1 = 1/10\ \alpha$, the upper limit in I_2 is essentially infinite, so that we get

$$I_2 \simeq \frac{1}{3}\log(250). \tag{3.20}$$

With these approximate values for I_1 and I_2 in equation (3.18), the critical temperature with inclusion of damping effects is given by

$$kT_c = (250)^{\frac{1}{3}}(1.14)\,\mathcal{E}_1 \exp\left[-1/N(0)V\right]. \tag{3.21}$$

The ratio of T_c to T_c^o is then

$$T_c/T_c^o = (250)^{1/3}(\mathcal{E}_1/\hbar\omega_c). \tag{3.22}$$

For $\epsilon_1 = 1/10\ \alpha = E_F/20$ and $\hbar\omega_c \simeq E_F$,

$$T_c/T_c^o \simeq 0.32, \tag{3.23}$$

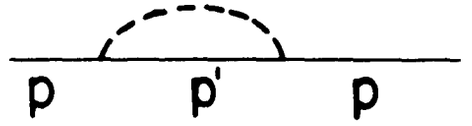
indicating that the effects of damping in our simplified treatment would reduce the critical temperature by a factor of about three. For the Emery and Sessler value of $T_c^o = 0.05^oK$, damping effects reduce the critical temperature to $T_c = 0.016^oK$. Experiments have been performed at temperatures $T \geq 5 \times 10^{-3}{}^oK$ without the appearance of a superfluid transition, which might indicate that damping is not

as important in determining T_c as we had hoped. However, the reduction of T_c^o by a factor of three could be important if a lower value of T_c^o were obtained. We should also add at this point that our work was based on a simplified model in which the potential was assumed to be of the S-wave type; i.e., V_{pp}, contains no angular dependence. On the other hand, the work of Emery and Sessler is based on the assumption of a D-wave term in the potential. Therefore, it might not appear valid for us to use their result of $T_c^o \simeq 0.05^oK$ in our calculation. We have seen, however, that the effect of damping is essentially to reduce the amount of phase space over which the potential acts. For this reason the ratio T_c/T_c^o would not be expected to vary significantly with the choice of angular dependence for the potential, although in a more exact calculation of T_c a D-wave term would explicitly be included in the potential in the energy gap equation.

Fig. 1 Self-energy diagram used to evaluate the integral equation for the energy gap Δ_p.

Fig. 2 Contour C_1 along which the p_0' integration is carried out in the energy gap equation (2.13).

Fig. 3 For an energy independent potential, $V_{p',p}$, the contour C_1 may be deformed to C_2 leading to the form (2.14) for the energy gap equation.


p
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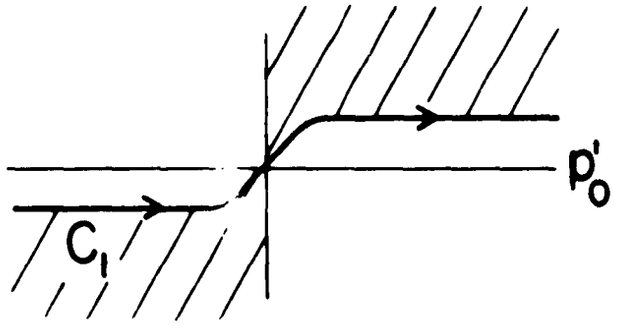
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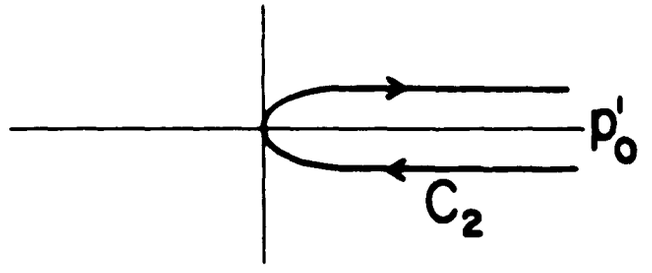
C1


p'0
C2

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